INTEREST

AND

BOND VALUES

BY

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PREFACE.

This little book is not to be regarded as a treatise on the Mathematical Theory of Interest—that theory which has been so ably expounded by Mr. King in *The Theory of Finance* and by Mr. Todhunter in the *Text Book of the Institute of Actuaries, Part I*. It is merely an explanation of the Interest Tables and Tables of Bond Values now in common use, and an attempt to instruct students concerning them.

While an elementary knowledge of Algebra is a powerful aid to the intelligent appreciation and use of such tables, and Algebra has not been excluded from the following pages, yet it is believed that nearly everything contained therein may be followed by anyone who will take the trouble to learn the meanings of the standard interest symbols.

Mainly written for the use of the author's own classes in the elementary mathematics of finance, it is hoped that the book may also be of value, not only to actuarial students but also to that increasing number of men who are finding it a business necessity to thoroughly understand the tables referred to.

THE UNIVERSITY,
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ERRATA.

p. 12, l. 14—for "is just half" read 'is about half'

p. 13, l. 30—for "quotes *a d*" read 'quotes a *d*'

p. 20, the last three lines should read as follows:—

$$`= \$100 \frac{1-v^5}{(1+i)^{1/4}-1} \text{ at } 4\% = \$1,807.22$$

$(1+i)^{1/4}$ may be obtained either by logarithms or by expansion by the binomial theorem'

p. 37, l. 20—for "$a_{\overline{n+n}|}$" read '$a_{\overline{n+m}|}$'

p. 39, ll. 18 and 20—for "$202.25" read '$201.12' , hence

l. 21—for "- $678.73 = $9,321.27" read '- $712.45 = $9,287.55'

l. 22—for "$202.25 $a_{\overline{50}|}$ at 2¼% = $6,034.01" read '$201.12 $a_{\overline{50}|}$ at 2¼% = $6,000.29'

l. 24—for "$9,321.27" read '$9,287.55'

l. 27—for "$67.13" read '$33.42'

p. 42, l. 20—for "$2,836.44" read '$7,525.44' hence

l. 21—for "$7,163.56" read '$2,474.56'

l. 22—for "$7,163.56 = $7,475.02" read '$2,474.56 = $2,582.16'

l. 23—for "$10,311.46" read '$10,107.60'

p. 46, l. 15—for "4% bonds" read '5% bonds'

p. 50, ll. 6 and 7 should read as follows:—

'The first method gives $V_o + \frac{j}{n}\left(1 - \frac{n-1}{n}\, i\right)$

$$= V_o + \frac{j}{n} - \frac{n-1}{n^2}\, i j\text{'}$$

p. 53, l. 21—for "greatest half-way" read 'greatest about half-way'

p. 84, l. 19—for "100 acres a year" read '1000 acres a year'

p. 88, Q. 13—for "in terms of *n*, *x* and *y*" read 'in terms of *x* and *y*'

INTEREST AND BOND VALUES.

CHAPTER I.

Interest and Discount.

We are all aware of the fact that men and corporations of undoubted ability to pay may generally be found who are willing to pay more than a dollar at some future date in return for a dollar today. The excess payment made when the borrowed dollar is returned is called *interest.* We are all equally aware of the corresponding fact that banks and similar institutions will give something less than a dollar today for a good promise to pay a dollar at some future date. The "something less" differs from the dollar by what is called *discount.*

Interest is quoted at so much per cent. per annum, is calculated on the sum lent, and is payable at the end of the year or at the ends of such sub-divisions of the year as may be agreed upon.

Discount is quoted at so much per cent. per annum, is calculated on the sum to be paid in the future, but is itself always payable in advance.

These are facts of common knowledge. Our theory of interest is based on these facts and has nothing whatever to do with the speculations of the Economist who searches for the reasons for these facts.

Interest calculations must be as old as civilization. There were money lenders in Thebes and Babylon. Nowadays such calculations commonly occur all over the world. It is therefore not surprising that a world wide system of interest

symbols should have been developed. The elements of this notation are as follows:—

i is the interest on 1 for one period (say a year).
1 at interest for one period will amount to $1+i$.
1 at interest for two periods will amount to $(1+i)^2$.
1 at interest for three periods will amount to $(1+i)^3$.
&c. &c. &c. &c.
1 at interest for n periods will amount to $(1+i)^n$.

Thus if the rate of interest be 5% per annum, $i=.05$ or .05 is the interest on 1 for one year.

1 at interest for one year will amount to 1.05.
1 at interest for two years will amount to $(1.05)^2=1.10250$.
1 at interest for three years will amount to $(1.05)^3=1.15763$.
1 at interest for n years will amount to $(1.05)^n$.

Again,
v is the present value of 1 due at the end of one period.
v^2 is the present value of 1 due at the end of two periods.
v^3 is the present value of 1 due at the end of three periods.
&c. &c. &c. &c.
v^n is the present value of 1 due at the end of n periods.

Since 1 is the present value of $1+i$ due at the end of one period, and v is the present value of 1 due at the end of one period,

therefore $1 : v :: 1+i : 1$, or $v(1+i)=1$.

$$i.e.\ v=\frac{1}{1+i} \text{ and } 1+i=\frac{1}{v}.$$

Thus if the rate of interest be 5% per annum

$$v=\frac{1}{1+i}=\frac{1}{1.05}=.95238, \text{ so that}$$

$v=.95238$ is the present value of 1 due one year hence.
$v^2=.90703$ is the present value of 1 due two years hence.
$v^3=.86384$ is the present value of 1 due three years hence.
&c. &c. &c. &c.

$v^n=\dfrac{1}{(1.05)^n}$ is the present value of 1 due n years hence.

d is the discount on 1 due one period hence, and obviously d must $= 1-v$. So we have

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i} = iv.$$

The three symbols i, d, v, are related as shewn in the following schedule.

The Value of	in terms of		
	i	d	v
i	i	$\frac{d}{1-d}$	$\frac{1-v}{v}$
d	$\frac{i}{1+i}$	d	$1-v$
v	$\frac{1}{1+i}$	$1-d$	v

The ordinary interest tables, such as those published by Colonel Oakes or by Mr. Archer, give the values of $(1+i)^n$ and v^n for numerous values of i ranging from ¾ of 1% up to 10%, and for values of n ranging from 1 period up to 200 periods.

Typical extracts might be:—

n	Amount of 1 at interest for n periods. $(1+i)^n$				n
	1% or $i=.01$	2% or $i=.02$	3% or $i=.03$	4% or $i=.04$	
5	1.05101	1.10408	1.15927	1.21665	5
10	1.10462	1.21899	1.34392	1.48024	10
15	1.16097	1.34587	1.55797	1.80094	15
20	1.22019	1.48595	1.80611	2.19112	20

n	Present value of 1 due n periods hence v^n				n
	1% or $i=.01$	2% or $i=.02$	3% or $i=.03$	4% or $i=.04$	
5	.95147	.90573	.86261	.82193	5
10	.90529	.82035	.74409	.67556	10
15	.86135	.74301	.64186	.55526	15
20	.81954	.67297	.55368	.45639	20

It should be noted that many tables use the word "years" in place of the word "periods." This is unfortunate since interest is usually compounded more frequently than once a year, and the word "years" must be understood to mean "half years" or "quarters" as circumstances demand.

These tables are of course applicable to any currency. From the extracts given above we see that at 4% interest compounded yearly for 5 years,

1000 dollars will amount to 1,216.65 dollars, or
1000 pounds will amount to 1,216.65 pounds, or
1000 francs will amount to 1,216.65 francs.
But if interest be compounded half yearly,
then \$1000 will amount to \$1,218.99.
While if interest be compounded quarterly,
the \$1000 will amount to \$1220.19.

Similarly, a good promise to pay \$1000 fifteen years hence is worth \$555.26 if the rate of interest be 4% compounded yearly.

Such tables not only shew the amount to which \$1. will accumulate in any time at any rate of interest, and the present value of a \$1. due at any time in the future at any rate of interest, but will also, if entered inversely, answer questions similar to the following.

1. In what time will \$1. amount to \$1.75 at 3% compounded half yearly? From the 1½% table we see the answer to be a little less than 38 periods, i.e. 19 years.

2. At what rate % compounded quarterly will \$1. amount to \$2.50 in 18 years? From the values opposite 72 periods we find the periodic rate required to be about $1\frac{9}{32}$% i.e. $5\frac{1}{8}$% per annum.

To obtain a value of $(1+i)^n$ or of v^n when n lies beyond the range of the tables we have only to multiply together two or more values that are given. Suppose for example that we want the present value of \$1000 due 40 years hence at 4% compounded quarterly. It will be \$1000 v^{160} at 1%. If our tables do not go beyond 100 periods, we have $v^{160}=v^{80}\times v^{80}$, or $=v^{100}\times v^{60}$, or any pair of suitable factors. Now $v^{100}=.36971$, and $v^{60}=.55045$; therefore $v^{160}=.20351$ and the value required is \$203.51.

An inspection of the tables of $(1+i)^n$ will shew the truth of the common rule—To find the time (number of periods) in which money will double itself at interest, divide 70 by the rate per cent. per period.

The proof of this rule is quite simple by the aid of Napier's logarithms.

$$2=(1+i)^n$$

$$\therefore \log_e 2=n \log_e(1+i)$$

$$\text{or } n=\frac{.693}{i-\frac{i^2}{2}+\frac{i^3}{3}-\&c.}=\frac{.693}{i}+.35$$

very nearly.

The rule is almost exactly true for rates of between 1% and 2% per period, e.g. for 4% compounded half yearly or quarterly or for 3% compounded half yearly. For higher periodic rates the results of the rule are slightly too small, but even for 8% per period the error is only one quarter of a period, or 3 months if this rate be compounded yearly.

Although interest is always quoted at so much per cent. per annum, it is usually payable more frequently than once in each year. In interest calculations when interest is payable only once a year it is said to be compounded yearly, or compounded with yearly rests, or to be convertible yearly; but if the interest is payable twice or four times a year it is said to be compounded half yearly or quarterly.

When interest is compounded more frequently than once a year the result is to produce an "effective" rate in excess of the "nominal" or quoted rate. For example, if the nominal rate be 4% and interest be compounded half-yearly, $1 will amount to $1.02 at the end of six months and to $\$(1.02)^2 =$ $1.0404 at the end of the year; or a nominal rate of 4% compounded half yearly gives an effective rate of 4.04%. Similarly, if the nominal rate be 4% compounded quarterly $1. will amount to $\$(1.01)^4 = \$1.0406$ at the end of a year; or a nominal rate of 4% compounded quarterly gives an effective rate of 4.06%. In general, a nominal rate of j per annum compounded m times a year gives an effective rate of i where

$$\left(1+\frac{j}{m}\right)^m = 1+i.$$

The following table illustrates the effect of frequency of compounding.

Nominal Rate of Interest	Effective rate when compounded.			
	half-yearly	quarterly	monthly	daily
2½%	2.5156%	2.5235%	2.5288%	2.5315%
3%	3.0225%	3.0339%	3.0416%	3.0454%
3½%	3.5306%	3.5462%	3.5567%	3.5620%
4%	4.0400%	4.0604%	4.0742%	4.0811%
4½%	4.5506%	4.5765%	4.5940%	4.6028%
5%	5.0625%	5.0945%	5.1162%	5.1271%

Thus, if the rate be 3% per annum, and the sum at interest be $10,000.00,

Compounding yearly the interest will be $300.00 a year,
Compounding half yearly the interest will be $302.25 a year,
Compounding quarterly the interest will be $303.39 a year,
Compounding monthly the interest will be $304.16 a year,
Compounding daily the interest will be $304.54 a year.

Or, the maximum advantage which can be gained by frequent compounding is only $4.54 a year on a sum of $10,000.00 at 3% and practically half this advantage is gained by compounding twice a year, while only another quarter of it is secured by compounding four times a year. These proportions apply, as may be seen, to all ordinary rates of interest.

There is no difference in essence between capital and interest. Each is money. Just as capital may be the interest accumulations of the past, so interest paid to-day and invested to-morrow will become nominally capital. Invested capital grows by the operation of interest and the rate of growth is called the rate of interest. A rate of growth is usually quoted at so much per period, as for example, so much per annum; but growth is a continuous process—not proceeding by isolated jumps. To illustrate—when we speak of a population increasing at a uniform rate of 3.65% per annum, we do not mean that for each 1,000,000 at the beginning of the year 36,500 people were suddenly added at the end of the year; nor do we mean that 100 persons were added per day, for that would imply that 1,000,000 people were increased by 100 during the first day of the year, and that 1,036,400 people were only increased by 100 during the last day of the year, which would not be a uniform rate of increase. What we do mean is that the population was increasing uniformly, that the increase was something less than 100 per day during the first days of the year and something more than 100 per day during the last days of the year, but that 36,500 people were added during the whole year. The rough assumption of 100 per million per day is nearly true, but obviously it is not exactly true. Such uniform daily increase—100 per million per day—would result in an increase of about 37,170 per million per annum or nearly 3.72%.

Simple Interest.

In order to avoid difficulties of calculation it is assumed in practice that $\frac{1}{365}$ of a year's interest will accrue each day, or that if the rate be 3.65% per annum and the sum be $1,000,000

$\frac{3.65}{100}$

$36500

100 per da for 36

the interest will be \$100 per day. The result of this assumption is called "simple interest." It is of course equivalent to compounding at the date of the calculation and introduces the gain due to such compounding. To show the amount of the error introduced by the assumption of simple interest, let us say that the rate is 3% per annum compounded half-yearly, and that the sum at interest is \$1,000,000, for 73 days.

The simple interest is	\$6,000.00
The true interest is $\$1,000,000\left\{(1.015)^{\frac{2}{5}}-1\right\}$	5,973.21
An error in excess of	\$26.79

Simple interest always produces an error in excess, that is to say, simple interest always exceeds true interest. This error is a maximum when the number of days for which simple interest is calculated is just half the period of compounding, and the error is least when the number of days for which it is calculated is either very small or very nearly equal to the period of compounding.

Algebraically, if i be the periodic rate, the true interest on 1 for $\frac{1}{n}$ of a period is

$$(1+i)^{\frac{1}{n}}-1=\frac{i}{n}-\frac{n-1}{\underline{|2}}\left(\frac{i}{n}\right)^2+\frac{(n-1)\,(2n-1)}{\underline{|3}}\left(\frac{i}{n}\right)^3-\frac{(n-1)(2n-1)(3n-1)}{\underline{|4}}\left(\frac{i}{n}\right)^4+-\&c.$$

This is a rapidly converging series for all ordinary values of i, and the assumption of simple interest that its value may be represented by its first term does not introduce a serious error unless the amount at interest be very large.

Most of the prevalent confusion regarding simple interest arises from the Arithmetics, wherein it is usually assumed that simple interest is based on one theory and compound interest on another. Problems are even set in which the assumption is made that money paid as interest can earn no interest itself. This absurdity comes from the attempt to carry the idea of simple interest past a period of compounding, quite ignoring

the fact that simple interest is only a ready approximation to true interest for a period less than a period of compounding. As an approximation for a broken period it is excellent, readily calculated and very nearly accurate. To carry the method beyond a period of compounding is to misunderstand it

Discount.

It is usually said that discount is interest paid in advance: that is quite true, but it is not the whole truth. When a banker quotes a rate of discount he is quoting a rate at which he will sell money, that is to say, at which he will sell the immediate right to draw cheques. His rate of discount is so much per cent. per annum and is calculated on the sum guaranteed by his customer to be paid in the future, but the discount itself is always payable in advance. Also, the rate of *discount* charged by the banker is *not* the rate of *interest* paid by his customer. The Banker who discounts a three months bill for $1000.00 at 6% will credit his customer with $985.00. That is to say the customer borrows $985.00 and guarantees the bank $1000.00 in payment at the end of three months. This represents an interest rate of over 6.09%. It should be noted that the banker did not quote an *interest* rate of 6% but a *discount* rate of 6%. If now, the same customer wished to deposit money in the same bank, the bank would quote him a rate of *interest*, probably 3%.

Most banks quote both rates, i.e. a discount rate (d) at which they will turn a right to future money into a right to a smaller amount of present cash, and also an interest rate (i) at which they will accept cash and give a right to a larger amount of money in the future. The Bank of England quotes a d—a discount rate—but gives no interest on deposits i.e. does not quote an i at all. Owing to a failure to appreciate the fact that the Bank was quoting a discount rate (d) and not an interest rate (i), our school arithmetics contain a curious distinction between "Banker's discount" and "true discount." On the false supposition that the Bank was quoting an i, the schoolmaster truly calculated that the

discount rate should be $\frac{i}{1+i}$ and accused the bank of charging more than it was entitled to. Thus, say the bank discount rate was 4% i.e. $d=.04$, the schoolmaster assumed wrongly that the 4% was an *interest* rate, i.e., $i=.04$, and on this false assumption correctly deduced that d should be equal to $\frac{i}{1+i}=\frac{.04}{1.04}=.038462$. It will be remembered that the Bank of England was originally a Whig institution and as such was opposed by the Church—then, even more than now, strongly Tory. The Schoolmasters of the eighteenth century were usually clergymen and were the authors of the prototypes of our modern school text books. In sympathy with their party these clerical authors made disparaging comments on the Bank, and, jumping to the conclusion that the bank rate was an interest rate, they invented the distinction between true discount and banker's discount to shew how the Bank was overcharging its customers This prejudice has long since passed away, and modern arithmetics are written or edited by men who are often neither Tories nor members of the Church of England, yet the old misconception has been copied and recopied—a wonderful example of the persistence of formalism.

Just as Simple Interest gave a ready approximation to true interest for a broken period, so with discount calculations the same approximation is used. If the rate of discount be d per period, the discount on 1 for $\frac{1}{n}$ of a period is practically taken as equal to $\frac{d}{n}$. The true discount of course would be

$$1-(1-d)^{\frac{1}{n}}=\frac{d}{n}+\frac{n-1}{\underline{\lvert 2}}\left(\frac{d}{n}\right)^2+\frac{(n-1)(2n-1)}{\underline{\lvert 3}}\left(\frac{d}{n}\right)^3$$
$$+\frac{(n-1)(2n-1)(3n-1)}{\underline{\lvert 4}}\left(\frac{d}{n}\right)^4+\ldots.$$

This is a rapidly converging series and may, without serious loss of accuracy, be represented by its first term unless the amount at discount be large. The error is always an error in

$e^{\delta}=(1+i)=(1-d)^{-1}$

defect, i.e. simple discount is always less than true discount. The error is, like that of simple interest, a maximum when the number of days for which discount is calculated is half the period for which the rate is quoted. To illustrate the amount of the error let us say that the discount rate is 5% per annum and that the sum subject to this rate is \$1,000,000, due 73 days hence.

True discount is \$1,000,000 $\left\{1-(.95)^{\frac{1}{5}}\right\}$ = ...	\$10,206.22
Simple discount is	10,000.00
An error in defect of	\$206.22

Bankers may be, doubtless are, guilty of many sins from their customers' point of view; but so far as the rates of interest and discount are concerned, the banker gives too much interest and charges too little discount on every transaction covering a period less than that for which his rates are quoted.

CHAPTER II.

Periodical Payments.

In the previous chapter reference was made to well known interest tables in which are published the values of such interest functions as $(1+i)^n$ and v^n for various values of i and n. The tables also usually contain two other interest functions applicable to periodical payments. These are

$s_{\overline{n}|}$ = the accumulated value of a series of n past payments of 1 each, made on the dates at which interest was compounded, the last payment having just been made.

$a_{\overline{n}|}$ = the present value of a series of n future payments of 1 each, to be made on the dates at which interest will be compounded, the first payment to be made one interest period hence.

These are commonly referred to as

$s_{\overline{n}|}$ = the amount of 1 per period.

$a_{\overline{n}|}$ = the present value of 1 per period.

But the assumptions of the above definitions must be understood in detail.

To express $s_{\overline{n}|}$ in terms of i and n:—

The payment just made is of course worth 1.

The payment made a period ago is now worth $(1+i)$.

The payment made two periods ago is now worth $(1+i)^2$.

&c. &c. &c. &c.

The first payment made $(n-1)$ periods ago is now worth $(1+i)^{n-1}$

$$\text{or } s_{\overline{n}|} = 1+(1+i)+(1+i)^2+\ldots\ldots+(1+i)^{n-1}$$

$$= \frac{(1+i)^n-1}{i}, \text{ the sum of the geometric series.}$$

$$\text{Or } (1+i)^n - 1 = i\, s_{\overline{n}|}.$$

This is a statement, the truth of which may be easily seen without algebra.

If A lent 1 to B n periods ago at rate i per period and B has made no periodical payments of interest, then B owes A $(1+i)^n$ today. Had B however paid i at the end of each period making his last payment today, he would owe A only the original 1, the capital of the loan. The difference $(1+i)^n-1$ must be the accumulated value of the interest payments of i each period, i.e. $i\ s_{\overline{n}|}$.

Or $(1+i)^n-1=i\ s_{\overline{n}|}$.

Reference to the tables will show that the first value of $s_{\overline{n}|}$, i.e. $s_{\overline{1}|}$, for any rate of interest is 1. The second value will be $1+(1+i)$ or at 2%, $1+1.02=2.02$.

at 5%, $1+1.05=2.05$.

Obviously the table of $s_{\overline{n}|}$ may be constructed from the table of $(1+i)^n$ by continuous addition.

It must be noted that the table is based on the assumption that the periodicity of the payments coincides with that of the compounding of the interest. For example, if the interest rate be 4% compounded half yearly, then payments of \$100 at the end of each quarter must be regarded as payments of \$201 at the end of each half year; while payments of \$100 at the end of each year must be regarded as payments of \$49.505 at the end of each half year. See the table of equivalent payments, page 30.

A few examples will illustrate the uses of the table of $s_{\overline{n}|}$.

1 The value of 10 payments of \$100 each, made annually during the past 10 years, the last payment having just been made, is, at 4% compounded yearly

$\$100\ s_{\overline{10}|}$ at $4\%=\$1200.61$.

2. The value of 10 annual payments of \$100 each, the last payment having been made 5 years ago is, at 5% compounded yearly,

$\$100\ (s_{\overline{15}|}-s_{\overline{5}|})$ at $5\%=\$1605.29$.

Or $\$100\ s_{\overline{10}|}\times(1+i)^5$ at $5\%=\ \$1605.29$.

3. The value of 20 semi-annual payments of \$100 each, the last payment having just been made, is, at 3% compounded half yearly,

$$\$100\ s_{\overline{20|}} \text{ at } 1\tfrac{1}{2}\% = \$2312.37.$$

4. The value of 20 quarterly payments of \$100 each, the last payment having just been made, is, at 4% compounded yearly, assuming simple interest for the broken periods,

$$\$(103+102+101+100)\ s_{\overline{5|}} \text{ at } 4\%$$
$$= \$406\ s_{\overline{5|}} \text{ at } 4\% = \$2199.03.$$

The error introduced by the assumption of simple interest for the broken periods is only 27 cents, the true value being \$2198.76.

$$\text{The true value is } \$100\left\{1+(1+i)^{\frac{1}{4}}+(1+i)^{\frac{2}{4}}+\ldots+(1+i)^{\frac{19}{4}}\right\}$$

$$= \$100\,\frac{(1+i)^5-1}{(1+i)^{\frac{1}{4}}-1} = \$2198.76.$$

$(1+i)^{\frac{1}{4}}$ may be obtained either by logarithms or by expansion by the binomial theorem.

5. To find the value of 20 yearly payments of \$100 each, the last payment having just been made, at $3\frac{1}{2}\%$ compounded half yearly.

We must first find the equivalent half yearly payment at this rate of interest. From the table on page 30 we see that the half yearly payment is \$49.566, so that the value required is \$49.566 $s_{\overline{40|}}$ at $1\frac{3}{4}\% = \$2836.87$. Or, without using the table of equivalent payments, we have the value required

$$= \$100\{1+(1+i)^2+(1+i)^4+(1+i)^6\ldots+(1+i)^{38}\} \text{ at } 1\tfrac{3}{4}\%$$

$$= \$100\,\frac{(1+i)^{40}-1}{(1+i)^2-1} = \$100\,\frac{s_{\overline{40|}}}{s_{\overline{2|}}} \text{ at } 1\tfrac{3}{4}\% = \$2836.87.$$

The table of $s_{\overline{n|}}$ may also be used inversely to answer questions similar to the following.

1. At what rate must annual deposits of \$1 each have been accumulated to amount to \$20 in 15 years, the interest having been compounded yearly and the last deposit having just been made? A reference to the values of $s_{\overline{15|}}$ will shew that the rate required is a trifle under 4%.

$\frac{v(1-v^n)}{1-v}$

2. In how many years at 3% compounded half yearly will deposits of \$1 each six months, amount to \$100? A reference to the $1\frac{1}{2}\%$ values of $s_{\overline{n}|}$ will shew that $s_{\overline{61}|} = 98.66$ and $s_{\overline{62}|} = 101.14$. This means that immediately after the 61st deposit the amount will be \$98.66 and six months' interest on this will bring it up to \$101.14. The required time is therefore 31 years, entailing 61 deposits in all.

To express $a_{\overline{n}|}$ in terms of i and n.

The present value of the first payment is v.

The present value of the second payment is v^2.

&c. &c. &c. &c.

The present value of the last payment is v^n.

Or $a_{\overline{n}|} = v + v^2 + v^3 + \dots\dots + v^n$.

$$= \frac{1-v^n}{i}, \text{ the sum of the geometric series.}$$

Or $1 = i\, a_{\overline{n}|} + v^n$.

This is a statement the truth of which may be easily seen without algebra.

If A lends B 1 today at rate i per period for n periods, B will make periodical payments of i and then a single payment of 1—the return of the capital.

The present value of the periodical payments of i is $i a_{\overline{n}|}$.

The present value of the repayment of the 1 is v^n.

The sum of these must equal the 1 that A lent B,

or $1 = i a_{\overline{n}|} + v^n$.

If n be infinite, that is to say if the loan is to last forever, there will be no repayment of the capital and

$$1 = i\, a_\infty \qquad \text{or } a_\infty = \frac{1}{i},$$

so that the value of a perpetuity of 1 is $\frac{1}{i}$.

Since $a_{\overline{n}|} = v + v^2 + v^3 + \dots\dots + v^n$.

$a_{\overline{1}|} = v$, $a_{\overline{2}|} = v + v^2$, and so on,

the $a_{\overline{n}|}$ table may obviously be constructed from the v^n table by continuous additon.

In using the $a_{\overline{n}|}$ table it should be noted that, just as with the $s_{\overline{n}|}$ table, the periodicity of the payments must coincide with that of the compounding of the interest. If the payments for which we need the table in any case do not so coincide we must use equivalent payments that do coincide. See the table on page 30.

A few examples will illustrate the use of the table of $a_{\overline{n}|}$.

1. The value of 10 annual payments of $100 each to be made during the next ten years, the first payment to be made one year from now, is, at 4% interest compounded yearly,

$$\$100\ a_{\overline{10}|} \text{ at } 4\% = \$811.09.$$

2. The value of 10 annual payments of $100 each, the first payment to be made 5 years hence, is, at 5% interest compounded yearly,

$$\$100\ (a_{\overline{14}|} - a_{\overline{4}|}) \text{ at } 5\% = \$635.27.$$

$$\text{Or } \$100\ a_{\overline{10}|} \times v^4 \text{ at } 5\% = \$635.27.$$

3. The value of 20 semi-annual payments of $100 each, the first payment to be made six months hence, is, at 3% compounded half yearly,

$$\$100\ a_{\overline{20}|} \text{ at } 1\tfrac{1}{2}\% = \$1716.86.$$

4. The value of 20 quarterly payments of $100 each, the first payment to be made three months hence, is, at 4% compounded yearly, assuming simple interest for the broken periods,

$$\$(103+102+101+100)\ a_{\overline{5}|} \text{ at } 4\%$$
$$= \$406\ a_{\overline{5}|} \text{ at } 4\% = \$1807.44.$$

The error introduced by the assumption of simple interest for the broken periods is only 22 cents, the true value being $1807.22.

$$\text{The true value is } \$100 \left(v^{\frac{1}{4}} + v^{\frac{2}{4}} + v^{\frac{3}{4}} + \ldots + v^{\frac{20}{4}} \right)$$

$$= \$100\ \frac{1 - v^5}{(1+i)^{\frac{1}{4}} - 1} \text{ at } 4\% = \$1807.22.$$

$(1+i)^{\frac{1}{4}}$ may be obtained either by logarithms or by the expansion of $(1+i)^{\frac{1}{4}}$ by the binomial theorem.

5. To find the value of 20 yearly payments of \$100 each, the first payment to be made one year hence, at $3\frac{1}{2}\%$ compounded half yearly.

We must first find what half yearly payment at this rate of interest is equivalent to \$100 a year.

From the table on page 30 we see that the equivalent half yearly payment is \$49.566, so that the value required is

$$\$49.566\, a_{\overline{40}|} \text{ at } 1\tfrac{3}{4}\% = \$1417.31.$$

Or, without using the table of equivalent payments, the value required is $\$100\,(v^2 + v^4 + v^6 + \dots + v^{40})$ at $1\frac{3}{4}\%$

$$= \$100\, \frac{1 - v^{40}}{(1+i)^2 - 1} = \$100\, \frac{a_{\overline{40}|}}{s_{\overline{2}|}} \text{ at } 1\tfrac{3}{4}\% = \$1417.31.$$

The tables of $a_{\overline{n}|}$ may also be used inversely to answer such questions as the following:—

1. At what rate of interest compounded half yearly will \$20 be the value of 25 future half yearly payments of \$1 each, the first payment to be made six months hence? A reference to the values of $a_{\overline{n}|}$ opposite 25 periods shews that the rate is nearly double $1\frac{13}{16}\%$ or nearly $3\frac{5}{8}\%$ per annum—more accurately $3\frac{19}{32}\%$ per annum.

2. For how many years will a cash payment of \$20 produce an annuity of \$1 a year at 4%, the annuity to be payable at the end of each year and interest to be compounded yearly? A reference to the 4% values of $a_{\overline{n}|}$ will shew that the answer is 41 years.

We can always find values for $s_{\overline{n}|}$ and $a_{\overline{n}|}$, when n lies outside the range of the tables we are using, by means of the formulae

$$s_{\overline{n+m}|} = s_{\overline{n}|} + (1+i)^n\, s_{\overline{m}|}$$

$$a_{\overline{n+m}|} = a_{\overline{n}|} + v^n\, a_{\overline{m}|}.$$

For example, if our tables run to only 50 periods

$$s_{\overline{95}|} = s_{\overline{50}|} + (1+i)^{50}\, s_{\overline{45}|}$$

$$a_{\overline{87}|} = a_{\overline{50}|} + v^{50}\, a_{\overline{37}|}.$$

The truth of such statements should be quite obvious.

The reciprocals of the functions $s_{\overline{n|}}$ and $a_{\overline{n|}}$ are also frequently tabulated and may be defined as follows:—

$\frac{1}{s_{\overline{n|}}} = s_{\overline{n|}}^{-1}$ = The sinking fund payment to be made at the end of each interest period in the future for n periods in order to accumulate to 1 n periods hence; the first sinking fund payment to be made one interest period from now and the last one n periods hence.

$\frac{1}{a_{\overline{n|}}} = a_{\overline{n|}}^{-1}$ = The future periodical payment to run for n periods that can be purchased by the payment of 1 now, the first of the periodical payments to come in at the end of one interest period from now.

These are commonly referred to as

$s_{\overline{n|}}^{-1}$ = Sinking fund required to produce 1.

$a_{\overline{n|}}^{-1}$ = Periodical payment which 1 will purchase.

But the assumptions of the above definitions must be understood in detail.

$$\text{Algebraically } \frac{1}{a_{\overline{n|}}} = \frac{i}{1-v^n} = \frac{i\,(1+i)^n}{(1+i)^n - 1},$$

$$\text{and } \frac{1}{s_{\overline{n|}}} = \frac{i}{(1+i)^n - 1}.$$

$$\therefore \frac{1}{a_{\overline{n|}}} - \frac{1}{s_{\overline{n|}}} = \frac{i\,(1+i)^n - i}{(1+i)^n - 1} = i,$$

$$\text{or } a_{\overline{n|}}^{-1} - s_{\overline{n|}}^{-1} = i.$$

This is a statement the truth of which may be seen without algebra. If A lends 1 to B now, B can repay principal and interest by n periodical payments of $a_{\overline{n|}}^{-1}$ each. Therefore $a_{\overline{n|}}^{-1}$ must consist of (1) the interest i on the 1 lent and (2) the sinking fund $s_{\overline{n|}}^{-1}$ needed to replace the 1 at the end of the n periods, i.e. $a_{\overline{n|}}^{-1} = i + s_{\overline{n|}}^{-1}$, or $a_{\overline{n|}}^{-1} - s_{\overline{n|}}^{-1} = i$.

Therefore it is not necessary to tabulate both $a_{\overline{n|}}^{-1}$ and $s_{\overline{n|}}^{-1}$.

Many tables contain only $a_{\overline{n}|}^{-1}$ from which any value of $s_{\overline{n}|}^{-1}$ can be found by inspection.

E.g. at $3\frac{1}{2}\%$ $a_{\overline{20}|}^{-1} = .07036108,$

$$i = .035.$$

$$\therefore s_{\overline{n}|}^{-1} = .03536108.$$

A few examples will illustrate the uses of these tables.

1. In order to produce a fund of \$10,000. at the end of 10 years by semi-annual sinking fund payments which will accumulate at 3% compounded half-yearly, there should be deposited at the end or each half year into this sinking fund

$$\$10{,}000\ s_{\overline{20}|}^{-1} \text{ at } 1\tfrac{1}{2}\% = \$432.46.$$

If deposits are to be made at the beginning of each half year $\$432.46 \div 1.015 = \$426.07$ will suffice. Or, directly from the tables, we have

$$\$10{,}000\ (s_{\overline{21}|} - 1)^{-1} = \frac{\$10{,}000}{23.4705} = \$426.07.$$

2. An investment of \$10,000 now will purchase semi-annual payments beginning six months hence and running for thirty years, on an interest basis of 4% compounded half yearly, amounting to \$10,000 $a_{\overline{60}|}^{-1}$ at $2\% = \$287.68$ each.

If the annuity is to be payable half yearly in advance the semi-annual payments will be only

$$\$287.68 \div 1.02 = \$282.04 \text{ each.}$$

Or, directly from the tables, we have

$$\$10{,}000(1 + a_{\overline{59}|})^{-1} = \frac{\$10{,}000}{35.4561} = \$282.04.$$

The same remarks regarding periodicity of payment and compounding apply of course to the tables of $s_{\overline{n}|}^{-1}$ and $a_{\overline{n}|}^{-1}$ as apply to those of $s_{\overline{n}|}$ and $a_{\overline{n}|}$.

To find the accumulated value of np payments of 1 each made during the past n years, the last payment having just been made, at a nominal rate of j per annum compounded q times a year.

Now a rate of j compounded q times a year is equivalent to a rate of h compounded p times a year where

$$\left(1+\frac{j}{q}\right)^q = \left(1+\frac{h}{p}\right)^p.$$

Therefore the accumulated value required is $s_{\overline{np}|}$ at rate $\frac{h}{p}$

$$= \frac{\left(1+\frac{h}{p}\right)^{np} - 1}{\frac{h}{p}} = \frac{\left(1+\frac{j}{q}\right)^{nq} - 1}{\left(1+\frac{j}{q}\right)^{\frac{q}{p}} - 1}.$$

Similarly the present value of np payments of 1 each to be made during the next n years, the first payment to be made $\frac{1}{p}$ of a year from now, at a nominal rate of j per annum compounded q times a year, is

$$a_{\overline{np}|} \text{ at rate } \frac{h}{p} \text{ where } \left(1+\frac{j}{q}\right)^q = \left(1+\frac{h}{p}\right)^p,$$

$$= \frac{1-\left(1+\frac{h}{p}\right)^{-pn}}{\frac{h}{p}} = \frac{1-\left(1+\frac{j}{q}\right)^{-qn}}{\left(1+\frac{j}{q}\right)^{\frac{q}{p}} - 1}.$$

If the periodical payments are increasing or decreasing by a fixed amount each period, that is to say, if the amounts of these payments form an arithmetical series, we can find either the accumulated value of such payments in the past or the discounted value of such payments to be made in the future. Suppose that we are dealing with past payments and that

x was deposited $n-1$ interest periods ago,
$x+y$ was deposited $n-2$ interest periods ago,
$x+2y$ was deposited $n-3$ interest periods ago,
&c. &c. &c. &c.
$x+(n-2)y$ was deposited 1 interest period ago,
and $x+(n-1)y$ was deposited today.

The accumulated value of these deposits now is

$$V = x\,(1+i)^{n-1} + (x+y)\,(1+i)^{n-2} + \ldots\ldots + (x+\overline{n-2}\,y)\,(1+i) + (x+\overline{n-1}\,y).$$

$$(1+i)\,V = x\,(1+i)^{n} + (x+y)\,(1+i)^{n-1} + \ldots\ldots + (x+\overline{n-2}\,y)\,(1+i)^2 + (x+\overline{n-1}\,y)\,(1+i).$$

$$\therefore i\,V = x\,(1+i)^{n} + y\left\{(1+i)^{n-1} + (1+i)^{n-2} + \ldots\ldots + (1+i) + 1\right\} - x - ny.$$

$$= x\left\{(1+i)^{n} - 1\right\} + y\,(s_{\overline{n|}} - n).$$

$$\text{or } V = x\,s_{\overline{n|}} + \frac{y}{i}\,(s_{\overline{n|}} - n).$$

Now suppose that we are dealing with future payments and that x will be received 1 interest period hence,

$x+y$ will be received 2 interest periods hence,

$x+2y$ will be received 3 interest periods hence,

&c. &c. &c. &c.

$x+\overline{n-2}\,y$ will be received $n-1$ interest periods hence,

$x+\overline{n-1}\,y$ will be received n interest periods hence.

The present value of these future payments is

$$V = xv + (x+y)v^2 + (x+2y)v^3 + \ldots\ldots (x+\overline{n-2}\,.\,y)v^{n-1} + (x+\overline{n-1}\,.\,y)v^{n}.$$

$$(1+i)\,V = x + (x+y)v + (x+2y)v^2 + \ldots\ldots (x+\overline{n-2}\,.\,y)v^{n-2} + (x+\overline{n-1}\,y)v^{n-1}$$

$$\therefore i\,V = x + y(v + v^2 + v^3 + \ldots + v^{n-1} + v^{n}) - (x+ny)v^{n}.$$

$$= x\,(1-v^{n}) + y\,a_{\overline{n|}} - nyv^{n}.$$

$$\text{or } V = x\,a_{\overline{n|}} + \frac{y}{i}\,(a_{\overline{n|}} - nv^{n}).$$

If the periodical payments form a geometrical series we can find expressions for either the accumulated value of such a series of payments made in the past, or the discounted value of such to be made in the future. Suppose that we are dealing with past payments and that

x was deposited n-1 interest periods ago,

xy was deposited n-2 interest periods ago,

xy^2 was deposited n-3 interest periods ago,

&c. &c. &c.

xy^{n-2} was deposited 1 interest period ago,

and xy^{n-1} was deposited today.

The accumulated value of these deposits now is

$$V = x(1+i)^{n-1} + xy(1+i)^{n-2} + xy^2(1+i)^{n-3} + \ldots \ldots + xy^{n-2}(1+i) + xy^{n-1}$$

$$= x\,\frac{(1+i)^n - y^n}{(1+i) - y}.$$

$$\text{Or } V = (1+i)^{n-1}\left\{x + x\left(\frac{y}{1+i}\right) + x\left(\frac{y}{1+i}\right)^2 + \ldots \ldots + x\left(\frac{y}{1+i}\right)^{n-2} + x\left(\frac{y}{1+i}\right)^{n-1}\right\}$$

$$= (1+i)^{n-1} . x . s_{\overline{n}|}$$

where $s_{\overline{n}|}$ is to be taken at rate j such that $1+j = \dfrac{y}{1+i}$.

$$\text{Or } V = (1+i)^{n-1} . x . (1 + a_{\overline{n-1}|})$$

where $a_{\overline{n-1}|}$ is to be taken at rate j such that

$$1 + j = \frac{1+i}{y}.$$

These alternatives enable one to avoid a negative rate of interest.

If we are dealing with future payments the discounted value of such a series is

$$V = xv + xy\,v^2 + xy^2\,v^3 + \ldots . xy^{n-2}\,v^{n-1} + xy^{n-1}\,v^n.$$

$$= x\,\frac{1 - y^n\,v^n}{(1+i) - y}.$$

$$\text{Or } V = \frac{1}{y}\left\{xvy + xv^2\,y^2 + xv^3\,y^3 + \ldots \ldots xv^{n-1}\,y^{n-1} + xv^n y^n\right\}$$

$$= \frac{x}{y} . a_{\overline{n}|}$$

where $a_{\overline{n}|}$ is to be taken at rate j such that

$$1 + j = \frac{1+i}{y}.$$

Or $V = v\{x + xyv + xy^2v^2 + \ldots\ldots xv^{n-2}y^{n-2} + xv^{n-1}y^{n-1}\}.$

$$= v.x.\grave{s}_{\overline{n}|}.$$

where $\grave{s}_{\overline{n}|}$ is to be taken at rate j such that

$$1+j=\frac{y}{1+i}\cdot$$

For example, the present value of a series of 50 annual payments beginning at \$100 one year hence and increasing by 2% per annum is, at 5% compounded yearly,

either $\$100\ \dfrac{1-(1.02)^{50}(1.05)^{-50}}{1.05-1.02}=\$100\ \dfrac{.7652835}{.03}=\$2{,}550.95$

or $\$\dfrac{100}{1.02}\, a_{\overline{50}|}$ at $2.94\% = \$2{,}550.95$

or $\$\dfrac{100}{1.05}\, s_{\overline{50}|}$ at $-2.86\% = \$2{,}550.95.$

To trace the growth of savings deposits intended for investment is interesting as well as important.

A fund is to be built up as follows—deposits of 1 made at the end of each period are to be accumulated at i per period for n periods. The accumulated amount is then to be invested in securities bearing j per period, $(j > i)$. During the next n periods the interest from the securities is to be deposited with the 1 each period for another n periods, and the accumulated amount again invested as before. This process is to be continued.

After n periods the fund will amount to $s_{\overline{n}|}$ at rate i, which we will write as plain $s_{\overline{n}|}$.

After $2n$ periods the deposits will amount to $(1+j\,s_{\overline{n}|})\,s_{\overline{n}|}$ and the whole fund to $s_{\overline{n}|}+(1+j\,s_{\overline{n}|})\,s_{\overline{n}|}$.

After $3n$ periods the deposits will amount to

$$\{1+j\,s_{\overline{n}|}+j(1+j\,s_{\overline{n}|})\,s_{\overline{n}|}\}\,s_{\overline{n}|}=(1+j\,s_{\overline{n}|})^2\,s_{\overline{n}|},$$

and the whole fund to $s_{\overline{n}|}+(1+j\,s_{\overline{n}|})\,s_{\overline{n}|}+(1+j\,s_{\overline{n}|})^2\,s_{\overline{n}|}$.

&c. &c. &c.

After rn periods the whole fund will amount to

$$s_{\overline{n}|}\{1 + (1 + j\, s_{\overline{n}|}) + (1 + j\, s_{\overline{n}|})^2 + \ldots\ldots (1 + j\, s_{\overline{n}|})^{r-1}\},$$

$$= s_{\overline{n}|}\frac{(1+j\, s_{\overline{n}|})^r - 1}{(1+j\, s_{\overline{n}|}) - 1} = \frac{(1+j\, s_{\overline{n}|})^r - 1}{j},$$

$$= \frac{(1+h)^{nr} - 1}{j} \text{ where } (1+h)^n = 1 + j\, s_{\overline{n}|},$$

$$= \frac{h}{j}\, \grave{s}_{\overline{nr}|} \text{ where } \grave{s}_{\overline{nr}|} \text{ is taken at rate } h.$$

So that the fund might have been built up by uniform deposits of $\frac{h}{j}$ to increase at rate h without any reinvesting.

Reference has already been made to the frequent necessity for finding the half yearly payment equivalent to a given yearly or quarterly payment, or vice versa.

A Table of equivalent payments at different periods is given below for ready reference. The method of obtaining the values may be illustrated, at 5% for example, as follows:

Quarterly payments of \$250 each are equivalent to half yearly payments of \$250+\$253.125=\$503.125.

Therefore half yearly payments of \$500 are equivalent to quarterly payments of $\frac{500.}{503.125}$ of \$250.=\$248.447.

Or, half yearly payments of \$500 are equivalent to quarterly payments of \$500 $s_{\overline{2}|}^{-1}$ at 1¼ %=\$248.447.

Quarterly payments of \$250 each are equivalent to yearly payments of \$250.+\$253.125+\$256.25+\$259.375=\$1018.75.

Therefore yearly payments of \$1000 are equivalent to quarterly payments of $\frac{1000.}{1018.75}$ of \$250=\$245.399, allowing no interest on interest throughout the year.

Or, yearly payments of \$1000 are equivalent to quarterly payments of \$1000 $s_{\overline{4}|}^{-1}$ at $1\frac{1}{4}\%$ = \$245.361 allowing for interest on interest throughout the year.

Half yearly payments of \$500 each are equivalent to yearly payments of \$500.+\$512.50=\$1012.50.

Therefore yearly payments of \$1000 are equivalent to half-yearly payments of $\frac{1000.}{1012.50}$ of \$500=\$493.827.

Or, yearly payments of \$1000 are equivalent to half-yearly payments of \$1000 $s_{\overline{2}|}^{-1}$ at $2\frac{1}{2}\%$=\$493.827.

Equivalent Periodical Payments.

Each payment made at the end of the period.
(Simple Interest).

Rate of Interest %	$1000 a year is equal to		$500 a half-year is equal to		$250 a quarter is equal to		Rate of Interest %
	Half-yearly paym'nts of	Quarterly paym'nts of	Yearly paym'nts of	Quarterly paym'nts of	Yearly paym'nts of	Half-yearly paym'nts of	
2.25	$497.20	$247.91	$1005.62	$249.30	$1008.44	$501.41	2.25
2.30	497.14	247.86	1005.75	249.28	1008.62	501.44	2.30
2.40	497.02	247.77	1006.00	249.25	1009.00	501.50	2.40
2.50	496.89	247.68	1006.25	249.22	1009.37	501.56	2.50
2.60	496.77	247.59	1006.50	249.19	1009.75	501.62	2.60
2.70	496.65	247.49	1006.75	249.16	1010.12	501.69	2.70
2.75	496.59	247.45	1006.87	249.14	1010.31	501.72	2.75
2.80	496.52	247.40	1007.00	249.13	1010.50	501.75	2.80
2.90	496.40	247.31	1007.25	249.10	1010.87	501.81	2.90
3.00	496.28	247.22	1007.50	249.07	1011.25	501.87	3.00
3.10	496.15	247.13	1007.75	249.04	1011.62	501.94	3.10
3.20	496.03	247.04	1008.00	249.00	1012.00	502.00	3.20
3.25	495.97	247.00	1008.12	248.99	1012.19	502.03	3.25
3.30	495.91	246.94	1008.25	248.97	1012.37	502.06	3.30
3.40	495.79	246.85	1008.50	248.94	1012.75	502.12	3.40
3.50	495.66	246.76	1008.75	248.91	1013.12	502.19	3.50
3.60	495.54	246.67	1009.00	248.88	1013.50	502.25	3.60
3.70	495.42	246.58	1009.25	248.85	1013.87	502.31	3.70
3.75	495.36	246.53	1009.37	248.83	1014.06	502.34	3.75
3.80	495.29	246.49	1009.50	248.82	1014.25	502.37	3.80
3.90	495.17	246.40	1009.75	248.79	1014.62	502.44	3.90
4.00	495.05	246.31	1010.00	248.76	1015.00	502.50	4.00
4.10	494.93	246.21	1010.25	248.73	1015.37	502.56	4.10
4.20	494.80	246.12	1010.50	248.69	1015.75	502.62	4.20
4.25	494.74	246.08	1010.62	248.68	1015.94	502.66	4.25
4.30	494.68	246.03	1010.75	248.66	1016.12	502.69	4.30
4.40	494.56	245.94	1011.00	248.63	1016.50	502.75	4.40
4.50	494.44	245.85	1011.25	248.60	1016.87	502.81	4.50
4.60	494.32	245.76	1011.50	248.57	1017.25	502.87	4.60
4.70	494.19	245.67	1011.75	248.54	1017.62	502.94	4.70
4.75	494.13	245.62	1011.87	248.52	1017.81	502.97	4.75
4.80	494.07	245.58	1012.00	248.51	1018.00	503.00	4.80
4.90	493.95	245.49	1012.25	248.48	1018.37	503.06	4.90
5.00	493.83	245.40	1012.50	248.45	1018.75	503.12	5.00
5.10	493.71	245.31	1012.75	248.42	1019.12	503.19	5.10
5.20	493.58	245.22	1013.00	248.39	1019.50	503.25	5.20
5.25	493.52	245.17	1013.12	248.37	1019.69	503.28	5.25
5.30	493.46	245.13	1013.25	248.35	1019.87	503.31	5.30
5.40	493.34	245.04	1013.50	248.32	1020.25	503.37	5.40
5.50	493.22	244.95	1013.75	248.29	1020.62	503.44	5.50
5.60	493.10	244.86	1014.00	248.26	1021.00	503.50	5.60
5.70	492.98	244.77	1014.25	248.23	1021.37	503.56	5.70
5.75	492.91	244.72	1014.37	248.22	1021.56	503.59	5.75
5.80	492.85	244.68	1014.50	248.20	1021.75	503.62	5.80
5.90	492.73	244.59	1014.75	248.17	1022.12	503.69	5.90
6.00	492.61	244.50	1015.00	248.14	1022.50	503.75	6.00

INTEREST AND BOND VALUES.

CHAPTER III.

The Straight Term Bond.

The straight term or ordinary coupon bond is a promise to pay a definite sum on a definite future date, and has attached to it a number of separate promises called coupons to pay interest in the meantime on the above sum at a specified rate. These coupons are meant to be cut off and presented for payment on the successive interest dates printed upon them. The first coupon to be cut off is usually dated six months after the issue date of the bond itself: the last coupon to be cut off being the one which bears the due date of the bond. These bonds are not subscribed for at a fixed price like stock, nor is the holder under any obligations such as are imposed upon the stock subscriber. They are bought as an investment to yield the purchaser a rate of interest appropriate to the circumstances of the bond-issuing corporation and the details of the issue. This *investment rate* has no connexion whatever with the *bond rate* at which the coupons are payable.

In arriving at a decision as to what price he is willing to bid for a bond, the purchaser should have in mind a number of considerations amongst which the following may be mentioned.

1. The extent and nature of the security mortgaged for the redemption of the issue.

2. The ability of the issuing corporation to earn the annual coupon payments.

3. The extent to which personal supervision of his investment may become necessary, and his capacity to exercise such supervision.

4. The breadth of the market in which his investment will be saleable in the event of need.

5. The current prices of apparently similar securities.

The examination of these matters results in the purchaser reaching a decision as to the rate of interest the investment should yield him under all the circumstances considered. Once that decision has been reached a mere reference to a book of Bond Values will indicate the price he can bid.

A sample extract from a Bond Table will shew the usual arrangement of values.

4% BOND.

with half yearly coupons.

Investment Rate % compounded half-yearly	Time to Maturity.			
	28 years	28½ years	29 years	29½ years
3.50	108.878	108.971	109.063	109.153
3.55	107.944	108.026	108.107	108.187
3.60	107.020	107.092	107.163	107.233
3.65	106.106	106.169	106.230	106.290
3.70	105.203	105.256	105.308	105.359

The sequence of values may be thus illustrated:—

Purchase price of a 4% bond with 29½ years to run bought to yield 3½%	$109.153
Add half a year's interest at 3½%	1.910
	111.063
Deduct value of coupon	2.000
Value of bond with 29 years to run	109.063
Add half a year's interest at 3½%	1.908
Forward	110.971

Brought forward	$110.971
Deduct value of coupon	2.000
Value of bond with 28½ years to run	108.971
Add half a year's interest at 3½%	1.907
	110.878
Deduct value of coupon	2.000
Value of bond with 28 years to run	108.878

Many tables of Bond Values have been published and some of them are probably quite accurate, but the writer has in his possession the thirteenth edition of a widely used table which contains over one hundred errors many of which are by no means trivial. The values given in any table should be checked before using, either by an *independent* table, or, better, by the use of the interest tables we have been discussing.

Consider a $10,000 bond, due exactly 20 years hence and bearing half yearly coupons at 5%. The holder of such a bond will receive

(i) $250. at the end of each half year for 20 years.

(ii) $10,000 at the end of 20 years.

To find the value of such a bond on an interest basis of 4½% compounded half yearly, we have only to discount the benefits to be received.

(i) $250 $a_{\overline{40}\vert}$ at 2¼%	$6,548.38
(ii) $10,000 v^{40} at 2¼%	4,106.46
	$10,654.84 in all.

Or we may argue as follows:—Had the bond borne coupons at 4½%, i.e. of the value of $225 each, its value would have been $10,000. But since the coupons are for $250 each, the value of the bond exceeds $10,000 by $25 $a_{\overline{40}\vert}$ at 2¼% = $654.84, or the value is $10,654.84.

And this is the price which an investor should pay to yield him 4½% compounded half yearly.

Immediately after the purchase the investment will be standing in the purchaser's ledger at its cost, \$10,654.84. Six months later a coupon will be due and will produce \$250. But the investment was made to yield 4½%. Therefore it should only be debited for interest with \$239.73 which is half a year's interest at the investment rate on the purchase price. The balance of the coupon, namely \$10.27 is a return of capital. So that after the cashing of the coupon the investment will stand at \$10,644.57on which the next half year's interest will be \$239.50 and the second coupon will contain \$10.50 of capital repaid. By continuing this process the investment will be written down little by little, but more and more, each half year until at the end of 19½ years and just after cashing a coupon it will stand at \$10,024.45 and the last coupon will consist of \$225.55 interest and \$24.45 capital, which with the \$10,000 then repaid will just balance the investment.

It may be worth noting that if the investor could find another investment for \$10.27 each half year to yield him 4½% compounded half yearly, he could leave the bonds in his ledger at their purchase price and, regarding \$239.73 as a uniform interest income from the bonds, place the \$10.27 in the other investment each half year. This second investment, if allowed to accumulate, would amount on the due date of the bond to \$10.27 $s_{\overline{40|}}$ at 2¼% = \$655.08 which together with the \$10,000 then payable would practically balance the purchase price.

Sometimes in connexion with trust funds it is not desirable to write the investment down, and it is usually impossible to find another investment for the small periodical repayments of capital at the same investment rate. In the purchase we have been considering it may be imperative that the capital should remain intact, and yet it may be impossible to invest a sinking fund at better than 3% compounded half yearly. Under such conditions the sinking fund will demand \$654.84 $s^{-1}_{\overline{40|}}$ at 1½% = \$12.07, in place of \$10.27 out of each coupon, thus reducing the investment yield to \$237.93 each half year on a capital of

\$10,654.84 which will remain intact This is equivalent to an investment rate of 4.466% instead of $4\frac{1}{2}\%$.

On the other hand suppose that the investor had bought \$10,000 of 4% bonds due 20 years hence and bearing half yearly coupons to yield him $4\frac{1}{2}\%$ compounded half yearly. A bond table will shew the price to be \$9345.16.

This should be checked by discounting the benefits as follows

$$\begin{aligned} \$200\ a_{\overline{40|}} \text{ at } 2\tfrac{1}{4}\% &= \$5{,}238.70 \\ \$10{,}000\ v^{40} \text{ at } 2\tfrac{1}{4}\% &= 4{,}106.46 \\ \hline & \ \$9{,}345.16 \text{ in all.} \end{aligned}$$

Or, we may argue that had the coupons been for \$225 each the value of the bond would have been at par. The coupons being only for \$200 each, the value will be at a discount of \$25 $a_{\overline{40|}}$ at $2\frac{1}{4}\%$ = \$654.84. The value is \$9345.16.

The first half year's interest should be $2\frac{1}{4}\%$ on the price, that is \$210.27 but the coupon would only produce \$200. Debiting the account with the interest due and crediting it with the coupon would write up the investment by \$10.27 to \$9,355.43. The next half year's interest would be \$210.50. So that the investment would have to be still further written up by \$10.50. Continuing this process we should find that six months before the due date of the bond, and immediately after the cashing of the penultimate coupon, the investment would be standing at \$9,975.55, on which the last half year's interest would be \$224.45. The last coupon would produce \$200 and the balance of the interest due would bring the investment up to the \$10,000. then payable.

Again, it is worth noting that if the investor could find someone to lend him \$10.27 each half year, on the security of his bonds, and allow such a series of little loans to accumulate at $4\frac{1}{2}\%$ compounded half yearly, he would, on the due date of his bonds owe this lender \$10.27 $s_{\overline{40|}}$ at $2\frac{1}{4}\%$ = \$655.08. So that the bonds might have remained on the purchaser's books at their purchase price, \$9,345.16, producing \$210.27 each half year; and when they were paid off, the difference be-

tween the \$10,000 and the purchase price would practically wipe out the accumulation of small loans effected to make up the interest.

Here also it may be not only undesirable to write up this investment but it may be important to maintain an annual income as large as possible consistent with the preservation of the capital; and yet small periodical borrowings such as we have indicated might be only possible at, say, 6%. Now the premium of \$654.84 at which the bonds will be repaid in excess of the purchase price will at this rate only permit of semi-annual borrowings for interest to the extent of \$654.84 $s^{-1}_{\overline{40}|}$ at 3% = \$8.68 each half year. By this means the investment will yield \$208.68 each half year on a uniform capital of \$9,345.16 or at the rate of 4.466% instead of 4½%.

It will be noticed that if the bond rate exceeds the investment rate of interest, the bond will be selling at a premium, but if the bond rate is less than the investment rate, the bond will be selling at a discount. The algebraic investigation of this premium or discount is instructive.

Consider a bond for 1 due n periods hence bearing coupons at j and selling at $1+p$, i.e. at a premium of p, to yield i.

$$\text{Then} \quad 1+p=j\, a_{\overline{n}|}+v^n \text{ at rate } i.$$
$$\text{But} \quad 1=i\, a_{\overline{n}|}+v^n \text{ at rate } i.$$
$$\text{Therefore} \quad p=(j-i)\, a_{\overline{n}|} \text{ at rate } i.$$

Or the premium is the present value of the excess interest in the coupons.

Similarly if j be less than i and the bond is selling at a discount of d we have

$$1-d=j\, a_{\overline{n}|}+v^n \text{ at rate } i.$$
$$1=i\, a_{\overline{n}|}+v^n \text{ at rate } i$$
$$\therefore\ d=(i-j)\, a_{\overline{n}|} \text{ at rate } i.$$

Or the discount is the present value of the shortage of interest in the coupons.

Moreover the Bond Tables might have been constructed from the interest table of $a_{\overline{n}|}$ without reference to that of $(1+i)^n$.

Examples:—\$10,000 of 20-year 5% bonds with half-yearly coupons, to yield 4% payable half yearly will sell at a premium of \$(250−200) $a_{\overline{40|}}$ = \$50 $a_{\overline{40|}}$ at 2% = \$1,367.77, or at a price of \$11,367.77.

Similar bonds with 3½% coupons sold on the same basis will be at a discount of \$(200−175) $a_{\overline{40|}}$ = \$25 $a_{\overline{40|}}$ at 2% = \$683.89, or at a price of \$9,316.11.

If a bond for 1 due n periods hence and bearing coupons at j sells for $1+p_n$ to yield i, and a similar bond due m periods hence sells for $1+p_m$ to give the same yield, we have

$$p_n=(j-i)\ a_{\overline{n|}} \text{ at } i; \text{ and } p_m=(j-i)\ a_{\overline{m|}} \text{ at } i.$$

Also p_{n+m}, the premium at which a similar bond due $n+m$ periods hence would sell, is $(j-i)\ a_{\overline{n+m|}}$ at i.

$$\begin{aligned}\therefore\ p_{n+m} &= (j-i)\ a_{\overline{n+m|}}\\ &= (j-i)\ \{a_{\overline{n|}}+v^n\ a_{\overline{m|}}\}\\ p_{n+m} &= p_n+v^n\ p_m.\end{aligned}$$

Similarly if a bond for 1 due n periods hence and bearing coupons at j sells for $1-d_n$ to yield i, we have

$$d_n=(i-j)\ a_{\overline{n|}} \text{ at } i; \text{ and } d_m=(i-j)\ a_{\overline{m|}} \text{ at } i$$

$$\begin{aligned}\text{and } d_{n+m} &= (i-j)\ a_{\overline{n+m|}}\\ &= (i-j)\ \{a_{\overline{n|}}+v^n\ a_{\overline{m|}}\}\\ &= d_n+v^n\ d_m.\end{aligned}$$

So that we may extend our bond tables to any number of periods beyond their present range. Suppose our tables only run to 50 years, and we want to find the value of a 65-year bond.

Let the investment rate be 4% compounded half yearly If the bond rate exceeds 4%, the 65-year bond will sell at a premium greater than that at which a 15-year bond will sell by the premium for which a 50-year bond would sell multiplied by v^{30} at 2%.

If the bond rate be less than 4%, the 65-year bond will sell at a discount greater than that at which a 15-year bond will sell by the discount for which a 50-year bond would sell multiplied by v^{30} at 2%.

We may also extend the Bond Table to other bond rates which are not given in the Table.

Important

Since the price of any bond is the present value of the capital plus the present value of the interest, discounting at the investment rate; the difference in value between two bonds for the same amount due on the same date depends entirely upon the difference between their coupons. Therefore if we keep the investment rate and the unexpired time uniform, the values of 3%, 4%, 5%, 6% &c. bonds will be a series in arithmetical progression, each term greater than the preceding term by the same amount. So that if values for only two bond rates are given we can find values for any other bond rates.

The price of a 3½% bond is exactly half the sum of the prices of similar bonds bearing 3% and 4% respectively. The price of a 3.65% bond exceeds that of a similar 3% bond by $\frac{65}{100}$ of the difference between the prices of the 3% and the 4% bond under the same conditions.

It is customary to consider yield rates as being compounded half yearly, and most bonds carry half-yearly coupons. The tables of bond values are made on the assumption that the bonds will carry such coupons, and that the investment or yield rates are to be compounded half yearly. But occasionally bonds bear yearly or quarterly coupons, and for purposes of comparison it is important that such bonds should be valued on an investment rate compounded half yearly. An example will illustrate this point.

Consider a 25-year, 5%, $10,000 bond with yearly coupons. To find the value of such a bond to yield the investor 4¼% compounded half yearly. The annual coupons of $500 each are at this investment rate equivalent to half yearly coupons of $247.37 each. (See page 30). Therefore the value of the bond is

$$\$10{,}000+\$(247.37-212.50)\ a_{\overline{50}|} \text{ at } 2\tfrac{1}{8}\%$$
$$=\$10{,}000+\$1{,}067.50=\$11{,}067.50.$$

Or $247.37 $a_{\overline{50}\vert}$ at 2⅛% =	$7,572.91
+$10,000 v^{50} at 2⅛% =	3,494.59
	$11,067.50.

Had an ordinary table of bond values been used to answer this question, ignoring the fact that the coupons were yearly and not half yearly as assumed in such a table, the result would have been $11,148.01 an error of $80.51 in excess.

The bond tables, however, might have been used as follows:

The value of a 5% bond with coupons of $250 each is		$11,148.01
The value of a 4½% bond with coupons of	$225 each is	10,382.67
Differences	$25	$765.34

Therefore a bond with coupons of $247.37 is worth

$$\$10{,}382.67 + \frac{22.37}{25} \text{ of } \$765.34 = \$11{,}067.50.$$

Similarly for a bond with quarterly coupons.

Consider a $10,000 bond due exactly 25 years hence and bearing quarterly coupons at 4%. To find the value of such a bond to yield 4½% compounded half yearly.

The quarterly coupons of $100 each are at the investment rate equivalent to half yearly coupons of ~~$202.25~~ each. 201.12

Therefore the value of the bond is

$$\$10{,}000 - \$(225.00 - \overset{201.12}{\cancel{202.25}})\, a_{\overline{50}|} \text{ at } 2\tfrac{1}{4}\%$$

$$= \$10{,}000 - \cancel{\$678.73} = \cancel{\$9{,}321.27}. \quad \$9{,}287.55$$

201.12 $712.45

$$\text{Or } \cancel{\$202.25}\, a_{\overline{50}|} \text{ at } 2\tfrac{1}{4}\% = \cancel{\$6{,}034.01} \quad 6000.29$$

$$\$10{,}000\, v^{50} \text{ at } 2\tfrac{1}{4}\% = 3{,}287.26$$

$$\cancel{\$9{,}321.27} \quad 9,287.55$$

Had an ordinary table of bond values been used in this case, ignoring the fact of the quarterly coupons, the result would have been $9,254.14, an error of ~~$67.13~~ in defect. $33.42

It may be worth noting that the use of an ordinary bond table in the case of a bond with yearly coupons will produce an error in excess which will be practically double the corresponding error in defect produced when such a table is used to value a similar bond bearing quarterly coupons.

Tables are published for bonds with yearly coupons, and others for bonds with quarterly coupons, but such tables quote yield rates compounding in agreement with the coupons, i.e. either yearly or quarterly. The investment yield compounded half yearly has however become almost an institution. It is therefore important to be able to value bonds with yearly or quarterly coupons on the basis of an investment yield compounded half yearly.

Bonds are sometimes repayable at a premium. Consider the case of a $10,000 bond bearing half yearly coupons at 6%, and repayable 15 years hence at a premium of 10%. What should be the price of such a bond to yield 5½% to the investor?

If we neglect the premium on redemption, the table of bond values will give $10,506.23
But the value of the premium is $1000 v^{30} at 2¾% = 443.15

Therefore the total value of the bond is ...... $10,949.38.

Or we might have regarded the bond as one for $11,000. bearing coupons at $5\frac{5}{11}$%. Then using the bond table:

$11,000 of 15-year 6% bonds are worth $11,556.86
$11,000 of 15-year 5% bonds are worth 10,443.14

Difference for 1% in bond rate is.......... $1,113.72
Difference for $\frac{5}{11}$ of 1% in bond rate is..... 506.24
$11,000 of 15-year $5\frac{5}{11}$% bonds are worth ...... $10,949.38.

Or still regarding the bond as one for $11,000 bearing coupons at $5\frac{5}{11}$%, it should sell at a discount equivalent to the shortage of interest in the coupons.

This discount will be $(302.50 − 300) $a_{\overline{30|}}$ at 2¾%.

= $2.50 $a_{\overline{30|}}$ = $50.62

The price therefore is $11,000—$50.62 = $10,949.38.

Consider the case of a \$10,000 bond bearing half yearly coupons at 5% for only the last 15 years of its term, and repayable 20 years hence at a premium of 25%. To find the value of such a bond to yield the investor $4\frac{3}{4}\%$ compounded half yearly. The bond tables are not directly applicable. Discounting the benefits by the use of interest tables we have

$$\$250\,(a_{\overline{40|}} - a_{\overline{10|}}) \text{ at } 2\tfrac{3}{8}\% = \$6{,}707.68$$
$$\$12{,}500\, v^{40} \text{ at } 2\tfrac{3}{8}\% = \quad 4{,}888.25$$
$$\$11{,}595.93.$$

The general theorem regarding the value of a loan has been established by the late Mr. Makeham of London as follows:

Let C be the capital repayable including any premium on repayment;

j be the nominal rate of interest payable on C or on any unpaid portion of C;

i be the lender's investment rate;

K be the present value of C at rate i;

while A is the purchase price of the loan.

Now A is the present value of C at rate i, i.e. K, plus the present value at rate i of the interest payments to be made at rate j.

Therefore the present value at rate i of the interest payments to be made at rate j is $A-K$.

Now had the loan borne a nominal rate i an investor would have paid par for it, i.e. A would have been equal to C. Therefore the present value at rate i of interest payments at rate i is $C-K$.

Therefore the present value at rate i of interest payments at rate j is $\frac{j}{i}\,(C-K)$.

So that $A = K + \frac{j}{i}\,(C-K)$, which is Makeham's formula.

We may illustrate this formula by applying it to an ordinary bond for 1 due n periods hence and bearing coupons at rate j.

In this case $C=1$, and $K=v^n$

Therefore the price is $v^n+\frac{j}{i}(1-v^n)$

$$=1-\left(1-\frac{j}{i}\right)(1-v^n).$$

A convenient formula for the calculation of bond values.

Since $1-\left(1-\frac{j}{i}\right)(1-v^n)=1-(i-j)\,a_{\overline{n}|}$

$$=1+(j-i)\,a_{\overline{n}|},$$

we have again deduced that a bond will sell at a premium or at a discount equal the present value of the difference between the coupons actually payable and those that would be payable under the yield rate.

Arithmetically Makeham's formula is of great value in cases where the capital is repayable by instalments. A series of bonds issued to repay capital by instalments is called a *Serial Issue.*

Consider a loan for \$10,000 at 6% payable half yearly, \$500 of the capital to be repayable with each payment of interest, and suppose we want to know the price to yield the investor 5¾% compounded half yearly.

In this case $C=\$10{,}000$.

$$K=\$500\ a_{\overline{20}|} \text{ at } 2\tfrac{7}{8}\%=\$2{,}836.44$$

$$C-K=\$7{,}163.56$$

$$\frac{j}{i}(C-K)=\tfrac{24}{23} \text{ of } \$7{,}163.56=\underline{\$7{,}475.02}$$

$$\therefore A=K+\frac{j}{i}(C-K)=\underline{\underline{\$10{,}311.46}}.$$

Again, consider a loan of \$10,000 at 6% payable half yearly to run for 10 years and then to be redeemed by yearly instalments of \$1000 at a premium of 10%, the last instalment of capital to be repaid 20 years after issue; and suppose we want to know the price of such a loan to yield the investor 5½% compounded half yearly.

In this case $C = \$11,000$ and $j = 5\frac{5}{11}\%$. $= \frac{600}{11,000}$

K = the present value of the instalments of \$1100 a year equivalent at the yield rate to \$542.54 each half year

$$= \$542.54\ (a_{\overline{40}|} - a_{\overline{20}|}) \text{ at } 2\tfrac{3}{4}\% \quad = \quad \$\ 4,801.94$$

$$C - K = \$6,198.06$$

$$\frac{j}{i}(C-K) = \tfrac{120}{121} \text{ of } \$6,198.06 \qquad = \quad \$\ 6,146.84$$

$$\therefore\ A = K + \frac{j}{i}(C-K) \qquad = \$\ 10,948.78.$$

There remain to be considered those bonds which are repayable by the operation of an accumulative sinking fund. The following is a case in point.

A foreign government issues a loan of \$1,000,000. at 5% with yearly coupons and agrees to set aside 6½% of the par value of the bonds each year to pay the coupons and for the redemption of the issue by annual drawings—the numbers on the bonds to be redeemed each year being chosen by lot. Here we have an annuity of \$65,000 a year devoted to the service of the loan until it is all redeemed. At the end of the first year \$50,000 will be needed to pay coupons, and \$15,000 of the bonds will be drawn and cancelled. At the end of the second year only \$49,250 will be needed for coupons and \$15,750 will be available for redemption purposes, and so on.

Since the \$1,000,000. must be the cash equivalent of the annuity of \$65,000, discounted at 5%, the bonds will be all paid off in n years where

$$1,000,000. = 65,000\ a_{\overline{n}|} \text{ on a } 5\% \text{ basis}$$

$$\text{or } a^{-1}_{\overline{n}|} = .065 \text{ in the } 5\% \text{ table.}$$

A reference to the table shews that n is a little over 30 years, which is of course quite independent of the issue price of the loan.

To find the amount of the issue that will be outstanding after any, say the 18th, annual drawing, we have only to find the cash value at the bond rate of the remaining payments of \$65,000 a year, that is \$65,000 $a_{\overline{12}|}$ at 5% = \$576,111.

To find the issue price of such a loan to net *the purchaser of the whole issue* $5\frac{1}{2}\%$ yearly, we have only to find the cash equivalent of \$65,000 a year for 30 years on a $5\frac{1}{2}\%$ basis, that is \$65,000 $a_{\overline{30}|}$ at $5\frac{1}{2}\% =$ \$944,693. The 30 years is slightly less than the true time, but a purchaser at $94\frac{1}{2}$ would be practically buying on a $5\frac{1}{2}\%$ basis.

The gamble introduced by the annual drawings makes it impossible to say what investment rate would be realised by the holder of any one bond of the series. Suppose that a purchaser of one bond who bought at 96. found that his bond was drawn for cancellation at the end of the first year. He would get \$105 after one year for an investment of \$96, that is $9\frac{3}{8}\%$. On the other hand if his bond was not drawn till the end of the 30 years he would only obtain an investment yield of about $5\frac{1}{4}\%$.

Loans of this character are sometimes complicated by the introduction of a temporary sinking fund to provide for redemptions less frequently than once a year.

Consider a loan of \$1,000,000 at 5% with half yearly coupons. For the service of this issue \$32,500 is to be set aside each half year, out of which the coupons are to be paid and the balance deposited in a sinking fund which accumulates at 4% compounded half yearly. At the end of every third year during the currency of the loan the sinking fund is to be applied to the redemption of bonds at a premium of 10%, the numbers on the bonds to be cancelled to be chosen by lot. To find in what time the issue will be redeemed by the operation of such a sinking fund.

In this case we may regard the issue as one of \$1,100,000 of bonds bearing $4\frac{6}{11}\%$.

The half yearly coupons will call for \$25,000, leaving \$7,500 for the sinking fund. The bonds redeemed each three years are purchased at par for the sinking fund, and the coupons on the bonds so purchased will increase the semi-annual contributions to that fund, which will grow as follows :—

End of 3rd year:—\$7,500 $s_{\overline{6}|}$ at 2%. (See page 27.)

End of 6th year:—\$7,500 $\{s_{\overline{6}|} + (1 + j\, s_{\overline{6}|})\, s_{\overline{6}|}\}$ where $j = 2\frac{3}{11}\%$

End of 9th year:—$\$7{,}500\ s_{\overline{6}|}\{1+(1+j\ s_{\overline{6}|})+(1+j\ s_{\overline{6}|})^2\}$

and so on.

End of 3*n*th year:—$\$7{,}500\ s_{\overline{6}|}\ \{1+(1+j\ s_{\overline{6}|})+(1+j\ s_{\overline{6}|})^2+\ldots$
$+(1+j\ s_{\overline{6}|})^{n-1}\}$

$$= \$7{,}500\ s_{\overline{6}|}\frac{(1+j\ s_{\overline{6}|})^n-1}{1+j\ s_{\overline{6}|}-1} = \$7{,}500\ \frac{(1+j\ s_{\overline{6}|})^n-1}{j}.$$

If the whole \$1,100,000. be then in the sinking fund we have

$$(1+j\ s_{\overline{6}|})^n - 1 = \frac{25{,}000}{7{,}500} = \frac{10}{3}$$

$$\therefore n = \frac{\log 4.3333}{\log(1+2\tfrac{3}{11}\%\text{ of } s_{\overline{6}|})} = \frac{\log 4.33333}{\log 1.14337}$$

$$= \frac{.6368221}{.0581867} = 10.9445.$$

Therefore the issue will be redeemed in 32.83 years: which means that the sinking fund will have more than enough money to redeem all the outstanding bonds at the eleventh drawing.

We might have finished the work without the aid of logarithms as follows:

We saw that n drawings were necessary where

$$(1+j\ s_{\overline{6}|})^n-1=\tfrac{10}{3}.$$

If we write $(1+h)^6$ for $1+j\,s_{\overline{6}|}$ which is 1.14337 we have $(1+h)^6=1.14337$.

A reference to the tables shews that h is a little over $2\tfrac{1}{4}\%$

$\therefore\ (1.0225)^{6n}-1=\tfrac{10}{3}$, or $(1.0225)^{6n}=4.3333$ at $2\tfrac{1}{4}\%$

Therefore $6n$ is a little below 66

or n a little below 11.

Or again, without much loss of accuracy we may neglect the fact that the coupons on bonds bought for the sinking fund will earn only 2% instead of $2\tfrac{3}{11}\%$ each half year for three years after purchase, and proceed as follows:—

The half yearly coupons on the whole loan call for \$25,000. This leaves \$7,500 for the sinking fund, which at the end of 3 years will amount to \$7,500 $s_{\overline{6}|}$ at $2\%=\$47{,}311$. The equivalent half yearly contribution which invested in the bonds

themselves at par would have produced the same result is $47,311 $s^{-1}_{\overline{6}|}$ at $2\frac{3}{11}\%$ = $7,449.

	$s^{-1}_{\overline{6}|}$	Δ	
At 2.250%	.15753	-49	$\therefore s^{-1}_{\overline{6}|}$ at 2.273% = .15744.
2.375%	.15704		

Therefore the bonds would be redeemed in the same time by uniform semi-annual payments of $25,000+7,449=$32,449. or they will all be redeemed in n half years where

$$\$1,100,000 = \$32,449\ a_{\overline{n}|} \text{ at } 2\tfrac{3}{11}\%$$

$$\text{where } a^{-1}_{\overline{n}|} \text{ at } 2\tfrac{3}{11}\% = .02950.$$

	$a^{-1}_{\overline{66}|}$	Δ	
At 2.250%	.02923	93	$\therefore a^{-1}_{\overline{66}|}$ at 2.273% = .02940.
2.375%	.03016		

Therefore the time required is a little less than 33 years.

Or again, regarding the issue as $1,000,000 of 4% bonds to be redeemed at 110, we might have assumed that the bonds were cancelled on redemption as follows:—

Writing C_0 for the million dollars of nominal capital, C_3, C_6, C_9 &c. for the capital outstanding at the end of 3, 6, 9 &c. years, we have

$$C_0 - \tfrac{10}{11}\,(32,500. - .025\ C_0)\ s_{\overline{6}|} \text{ at } 2\% = C_3$$

or writing p for 32,500, i for .025 and s for $s_{\overline{6}|}$ at 2%

$$\begin{aligned} & C_0 - \tfrac{10}{11}\,(p - i\ C_0)\ s = C_3 \\ & \text{or } C_0\ (1 + \tfrac{10}{11}\, i\ s) = \tfrac{10}{11}\ p\ s + C_3; \\ & \text{so } C_3\ (1 + \tfrac{10}{11}\, i\ s) = \tfrac{10}{11}\ p\ s + C_6 \\ & \therefore (C_0 - C_3)\ (1 + \tfrac{10}{11}\ i\ s) = C_3 - C_6; \\ & \text{so } (C_3 - C_6)\ (1 + \tfrac{10}{11}\ i\ s) = C_6 - C_9. \\ & \qquad \&c. \qquad\qquad \&c. \end{aligned}$$

That is to say the triennial redemptions of capital are increasing in geometrical ratio with a constant factor $(1 + \frac{10}{11}\,is)$. The first term in the series of redemptions is $C_0 - C_3$, which is $\frac{10}{11}(p - i\ C_0)s$. Therefore there must be n drawings before the whole loan is redeemed where

$$\tfrac{10}{11}(p-i\,C_0)s \times \frac{(1+\frac{10}{11}i\,s)^n-1}{1+\frac{10}{11}i\,s-1} = C_0$$

$$\text{or } (1+\tfrac{10}{11}i\,s)^n = 1+\frac{\frac{10}{11}\,i\,s\,C_0}{\frac{10}{11}(p-i\,C_0)s} = 1+\frac{i\,C_0}{p-i\,C_0}$$

in figures, $(1+\frac{10}{11}$ of $2\frac{1}{2}\%$ of $6.30812)^n = 1+\dfrac{25{,}000}{7{,}500}$

or $(1.14337)^n = 4.33333$

$$\therefore n = \frac{\log 4.33333}{\log 1.14337} = \frac{.63682}{.05819} = 10.94$$

or the time required is $3n$ years $= 32.8$ years.

The progress of this loan might be scheduled as follows:

3-year period	The semi-annual payment of $32,500.		Amount of Sink. fund at end of the period. (3) $\times s_{\overline{6}|}$ at 2%	Bonds redeemed at end of the period. (4) $\times \frac{10}{11}$	Total redemption to end of the period.	Amount of Bonds outstanding at end of the period.
	for Coupons	for Sink. fund				
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	$25,000.	$7,500.	$47,311.	$43,010.	$43,010.	$956,990.
2	23,925.	8,575.	54,094.	49,176.	92,186.	907,814.
3	22,695.	9,805.	61,849.	56,226.	148,412.	851,588.
4	21,290.	11,210.	70,716.	64,287.	212,699.	787,301.
5	19,683.	12,817.	80,854.	73,504.	286,203.	713,797.
&c	&c.	&c.	&c.	&c.	&c.	&c.

It should be noticed that each value in the 5th column may be obtained from the preceding value by increasing the latter by 14.337%.

It may reasonably be objected that bonds are almost always in fixed denominations and that fractional amounts could not be redeemed as is assumed in the above calculations.

In order to test the error introduced by this assumption let us say that the bonds are in denominations of $1000 each.

Then the actual working of the sinking fund would be as follows:

First period—\$7,500 $s_{\overline{6}|}$ = \$47,310.91, enough to redeem \$43,000 of the issue leaving \$957,000 outstanding.

Second period—\$10.91 $(1.02)^6$+\$8,575 $s_{\overline{6}|}$=\$54,104.43, enough to redeem \$49,000 of the issue leaving \$908,000 outstanding.

Third period—\$204.43 $(1.02)^6$+\$9,800 $s_{\overline{6}|}$ = \$62,049.81, enough to redeem \$56,000 of the issue leaving \$852,000 outstanding.

Fourth period—\$449.81 $(1.02)^6$ + \$11,200 $s_{\overline{6}|}$= \$71,157.52, enough to redeem \$64,000 of the issue leaving \$788,000 outstanding.

Fifth period—\$757.52 $(1.02)^6$ + \$12,800 $s_{\overline{6}|}$ = \$81,597,04, enough to redeem \$74,000 of the issue leaving \$714,000 outstanding against which there is a balance of \$197. in the sinking fund. That is to say a net debt of \$713,803. Our unpractical assumption resulted in shewing a net debt of \$713,797 at the same date: an error of only \$6. Practically, the sole cause of the error is the difference between the 5% bond rate and the 4% sinking fund rate on the small sums left between drawings to accumulate in the sinking fund.

So far we have been considering bonds bought either at issue or at a regular coupon date. But bonds are of necessity bought between coupon dates and when this happens the seller is obviously entitled to some portion of the value of the current coupon. There is no difficulty when the bond is sold at a flat price which is supposed to include the adjustment in question; nor should there be any difficulty where the bond is sold at a price "and interest" which means accrued bond interst and is easily ascertained. It is when the bond is sold on a yield basis that difficulties have arisen. There is a want of uniformity among bond dealers in this matter, although of recent years most of them, when dealing on a yield basis, are using the second of the methods to be now described.

If an ordinary coupon bond is sold between two coupon

dates to yield the purchaser a definite investment rate of interest, it is sold on a yield basis.

To find the price.

True Price = Value as at last coupon date, plus true interest for the fraction of a period elapsed since that date.

First method of approximation:

Price $= V_0 + I - D.$

where V_0 = Value as at last coupon date.

I = Simple interest on the face value of the bond at the bond nominal rate for the time since the last coupon date.

D = Banker's discount on I at the investment rate for the time to the next coupon date.

Second method of approximation:

Price $= V_0 + I - D.$

where V_0 = Value as at last coupon date.

I = Simple interest on V_0 at the investment rate for the time since the last coupon date.

D = Banker's discount on I at the investment rate for the time to the next coupon date.

Third method of approximation:

Price $= (V_1 + C) \div (1 + I)$

Where V_1 = Value as at next coupon date.

C = Face value of current coupon.

I = Simple interest on 1 at the investment rate for the time to the next coupon date.

Fourth method of approximation:

$$\text{Price} = V_0 + \frac{1}{n}(V_1 + C - V_0)$$

Where V_0, V_1 and C have the same values as before and $\frac{1}{n}$ = the fraction of a coupon period that has elapsed since the last coupon date.

All four methods are, or were recently, in actual use.

Discussing the problem algebraically we will consider a bond for 1 bearing coupons for j per period, sold $\frac{1}{n}$ of a period after a coupon date to yield the purchaser i per period.

If V_0 be the value at the last coupon date and V_1 be the value at the next coupon date we have

$$V_0(1+i)-j=V_1.$$

The true sale price is $V_0\,(1+i)^{\frac{1}{n}}$

$$=V_0\left(1+\frac{i}{n}-\frac{n-1}{2n^2}i^2+\frac{(n-1)\,(2n-1)}{6\,n^3}i^3-+\,\&c.\right).$$

The first method gives $V_0\left\{1+\frac{j}{n}\left(1-\frac{n-1}{n}\,i\right)\right\}$

$$=V_0\left\{1+\frac{j}{n}-\frac{n-1}{n^2}ij\right\}.$$

The second method gives $V_0\left\{1+\frac{i}{n}\left\{1-\frac{n-1}{n}\,i\right)\right\}$

$$=V_0\left\{1+\frac{i}{n}-\frac{n-1}{n^2}\,i^2\right\}.$$

The third method gives $(V_1+j)\left(1+\frac{n-1}{n}\,i\right)^{-1}$

$$=V_0\,(1+i)\left(1-\frac{n-1}{n}i+\frac{(n-1)^2}{n^2}i^2-\frac{(n-1)^3}{n^3}\,i^3+-\,\&c.\right)$$

$$=V_0\left\{1+\frac{i}{n}-\frac{n-1}{n^2}\,i^2+\frac{(n-1)^2}{n^3}\,i^3-+\,\&c.\right\}.$$

The fourth method gives $V_0+\frac{1}{n}\,(V_1+j-V_0)$

$$=V_0+\frac{1}{n}\{V_0(1+i)-V_0\}$$

$$=V_0\left(1+\frac{i}{n}\right).$$

Arithmetically we may take the following example \$1,000,000 of 4% bonds due 1 July 1940, with half yearly coupons 1 July and 1 January, bought on 1 September 1910 to yield 5% compounded half yearly.

True Price = (Tabular value as at 1 July 1910) $\times\,(1.025)^{\frac{1}{3}}$
= \$845,456.72 $\times$ 1.00826484
= \$852,444.28.

True Price $\times (1.025)^{\frac{2}{3}}$ = \$852,444.28 $\times$ 1.01659798
= \$866,593.14 = \$846,593.14 + \$20,000.00
= Tabular value as at 1 January 1911 + value of coupon then due.

First Method:

Tabular value as at I July 1910		\$845,456.72
Two months' interest at 4% on \$1,000,000	\$6,666.67	
Less 4 months' "discount" at 5% on \$6.666.67	111.11	6,555.56
Approximate price		\$852,012.28
Error in defect		432.00
True Price		\$852,444.28

Cost price as at 1 September, 1910	\$852,012.28
4 months' interest thereon at 5%	14,200.20
	\$866,212.48
Value of coupon due 1 January, 1911	20,000.00
Purchaser's value as at 1 January, 1911	\$846,212.48
Error in defect	380.66
Tabular value as at 1 January, 1911	\$846,593.14

Second Method:

Tabular value as at 1 July 1910		\$845,456.72
Two months' interest at 5% on \$845,456.72	\$7,045.47	
Less 4 months' "discount" at 5% on \$7,045.47	117.43	6,928.04
Approximate price		\$852,384.76
Error in defect		59.52
True Price		\$852,444.28

Cost price as at 1 September, 1910	$852,384.76
4 months' interest thereon at 5%	14,206.41
	$866,591.17
Value of coupon due 1 January, 1911	20,000.00
Purchaser's value as at 1 January, 1911	$846,591.17
Error in defect	1.97
Tabular value as at 1 January, 1911	$846,593.14

Third Method:

Tabular value as at 1 January 1911 with coupon .	$866,593.14
Four months' interest on $1. at 5% is $.01\dot{6}$	
Approximate price = $866,593.14 ÷ $1.01\dot{6}$	$852,386.69
Error in defect	57.59
True Price	$852,444.28

Cost price as at 1 September, 1910	$852,386.69
Four months' interest thereon at 5%	14,206.45
	$866,593.14
Value of coupon due 1 January, 1911	20,000.00
Tabular value as at 1 January 1911 (no error) . . .	$846,593.14

Fourth Method:

Tabular value as at 1 July 1910	$845,456.72
Two months' interest thereon at 5%	7,045.47
Approximate price	$852,502.19
Error in excess	57.91
True Price	$852,444.28

Cost price as at 1 September, 1910	$852,502.19
Four months' interest thereon at 5%	14,208.37
	$866,710.56
Value of coupon due 1 January, 1911	20,000.00
Purchaser's value as at 1 January, 1911	$846,710.56
Error in excess	117.42
Tabular value as at 1 January, 1911	$846,593.14

In the first method of approximation the underlying argument seems to be that the only change in value from one coupon date to the next is due to the accrued bond interest. This is of course a serious error unless the bond rate is close to the investment rate. The method produces an error in defect when the bond rate is less than the investment rate and vice versa: this error increases with the time since the last coupon date. The method probably owes its origin to a confusion between selling on a yield basis and selling at a price and interest.

In the second method of approximation the underlying argument is sound, but the attempt to correct the error of simple interest can hardly be called successful. The method always produces an error in defect. The error is roughly equal to half the "discount" item, and it is greatest half way between two coupon dates.

In the third method of approximation the underlying argument contains the error of "simple interest". The method always produces an error in defect which is uniformly slightly less than the error of the second method.

In the fourth method of approximation the underlying argument contains the error of "simple interest". The method always produces an error in excess nearly equal in amount to the opposite error of the second method.

The advantage of the error in defect is that it will be more or less compensated by the error that the purchaser will make at the next coupon date when he adds simple interest for the

time he has held the bond. The third approximate method is such that this "compensation" is complete.

The second approximation is, as already mentioned, the one in common use in Canada today; yet it is frequently used with a bond table to only two places of decimals. The result of using so "short" a table is to introduce initially an error which may be as great as

	\$50.00 on	\$1,000,000.00	of bonds
or	5.00 on	100,000.00	of bonds
or	.50 on	10,000.00	of bonds.

Not a very serious error, but one which renders the subsequent items of "interest" and "discount" carried out to the nearest cent an absurdity. This lack of "the sense of the fitness of things" may be seen on almost any statement of any bond-dealer who sells on a yield basis. If the bond value used be true to the nearest \$1.00, then the interest and discount items should also be carried out only to the nearest \$1.00.

It is usual for the coupon dates to correspond with the due date of the bond. For example if the bond be due on a 5th May, the coupons are usually due on 5th May and 5th November in each year. But such a correspondence is not always observed. Consider the following case which is not an imaginary one.

Bonds for \$100,000 at 4% due on 1 January 1930, had half yearly coupons dated 1 May and 1 November in each year, and were sold on 1 March 1910 to yield the purchaser 5% compounded half yearly. To find the price, we will use the second method of approximation referred to.

With the capital there must be payable coupons for two months' bond interest, that is, \$666.67.

The value as at 1 January 1910 of the capital and these coupons is \$100,666.67 v^{40} at $2\frac{1}{2}\%$ = ...		\$37,491.
Add two months' interest	\$312.4	
less four months' discount	5.2	307.
Value of Capital and last coupons as at 1 March 1910		\$37,798.

The value of all the other coupons as at 1 November 1909 is \$2,000 $a_{\overline{40}\mid}$ at $2\frac{1}{2}\%$		\$50,206.
Add four months' interest	\$836.8	
less two months' discount	7.0	830.
Value of the other coupons as at 1 March, 1910		\$51,036.
Total value of bond as at 1 March, 1910		\$88,834.

The inverse problem—given the price to find the yield—is practically of very great importance and while the interest and bond tables available gave investment rates only at considerable intervals the problem was by no means a simple one. The best results were given by the formulae of Finite Differences. Now that bond and interest tables are published for investment rates proceeding by one twentieth of 1% the problem has practically lost its difficulty.

Consider a 4% 40-year bond which sells at $93\frac{1}{2}$. The Bond tables shew that such a bond

would yield 4.35% if sold at 93.39;
or yield 4.30% if sold at 94.30.

That is an increase of .91 in the price produces a drop of .05 in the yield.

Therefore a price of 93.50 corresponds to a yield of $4.35 - .006 = 4.344\%$.

If we know the price at which any loan is issued we can use Makeham's formula to find a close approximation to the yield.

Changing the formula algebraically into the form

$$i = j + \frac{C-A}{C-K}\, i \text{ will give the best results.}$$

To use this formula, we make the best guess we can at i and interpret the right hand side of the equation in accordance with that guess. The result should be very near the truth.

Suppose that a 5% debenture with half yearly coupons and redeemable at 105 in 20 years is sold at 113.67. What will be the yield to the purchaser ?

4% will be a fair guess.

At this rate $K=105\ v^{40}$ at $2\%=47.55$

$C=105\ :\ C-A=-8.67\ :\ C-K=57.45$

$$j=\frac{5}{105}=.04762.$$

$$\therefore i=.04762-\frac{8.67}{57.45}\times .04$$

$$=.04762-.00604=.04158 \text{ or } 4.158\%.$$

Our guess was too low. Let us try 4¼%.

At this rate $K=105\ v^{40}$ at $2\tfrac{1}{8}\%=45.28$.

$C-A=-8.67$ and $j=.04762$ as before.

$C-K=59.72$.

$$\therefore i=.04762-\frac{8.67}{59.72}\times .0425$$

$$=.04762-.00617=.04145 \text{ or } 4.145\%.$$

A trial rate of 4% produced 4.158%.

A trial rate of 4¼% produced 4.145%.

∴ the true rate is $(4+x)$ % where

$x : \tfrac{1}{4} :: .158-x : .013$ or $x=.150$.

The required rate is 4.150%.

A readier method of finding the yield would be as follows: Since the bond, bought at 113.67, will be redeemed at 105, there will be a loss of 8.67 on redemption. To make good this loss there should be set aside out of each coupon a sum equal to 8.67 $s^{-1}_{\overline{40|}}$ at the yield rate. Taking 4% as our first guess at this yield, we have 8.67 $s^{-1}_{\overline{40|}}$ at $2\%=.1435$. But each coupon is worth 2.5. This leaves 2.3565 as a uniform half yearly return on a capital of 113.67 which will remain intact; that is, a yield of 4.146%. Trying 4¼% we have 8.67 $s^{-1}_{\overline{40|}}$ at $2\tfrac{1}{8}\%=.1397$. This leaves 2.3603 as the half yearly return on the unimpaired 113.67 ; that is, a yield of 4.153%.

A trial rate of 4% produced 4.146%.

A trial rate of 4¼% produced 4.153%.

Therefore the true rate is $(4+x)\%$,

where $x : .25 :: x - .146 : .007$, or $x = .15$, and the required rate is 4.15%.

When neither a bond table nor an interest table is available a fairly close approximation to the rate may be obtained by an algebraic formula.

Consider a bond for 1 bearing coupons at j and due n periods hence, bought at $1+p$. To find i, the yield rate.

Now $p = (j-i)\, a_{\overline{n|}}$ at rate i.

or $\dfrac{p}{j-i} = \dfrac{1-v^n}{i} = \dfrac{1-(1+i)^{-n}}{i} = n\left(1 - \dfrac{n+1}{2}\, i\right)$ approximately.

$\therefore \dfrac{n\,(j-i)}{p} = \left(1 - \dfrac{n+1}{2}\, i\right)^{-1} = 1 + \dfrac{n+1}{2} i$ approximately,

$$\text{whence } i = \frac{2\,(jn-p)}{n(2+p)+p}.$$

Example: A 4% bond with half yearly coupons, and having 25 years to run is bought at 113%. To find the yield.

Here $j = .02$; $n = 50$; $p = .13$

$$\therefore i = \frac{2\,(1-.13)}{50 \times 2.13 + .13} = \frac{1.74}{106.6} = .0163,$$

or a yield of 3.26% per annum.

The true yield is 3.24 % per annum.

If the bond be sold at a discount, the formula becomes

$$i = \frac{2\,(jn+d)}{n(2-d)-d}.$$

Example:—A 4% bond with half yearly coupons, and having 25 years to run is bought at 92½%. To find the yield.

Here $j = .02$; $n = 50$; $d = .075$

$$\therefore i = \frac{2\,(1+.075)}{50 \times 1.925 - .075} = \frac{2.15}{96.175} = .02235,$$

or a yield of 4.47% per annum.

The true yield is 4.50%.

The values of the interest functions v^n and $a_{\overline{n|}}$ may readily be deduced for all the investment rates of interest and all the unexpired terms of the bonds scheduled in the bond table.

For example the bond table contains the following:

Investment Rate	Bond Rate	Unexpired term	Price
3.60%	4%	40 years	108.444650

which means that at 1.8%

$$100\ v^{80}+2\ a_{\overline{80|}}=108.444650$$

$$\text{but}\quad 100v^{80}+1.8\ a_{\overline{80|}}=100.$$

$$\therefore .2\ a_{\overline{80|}}=8.444650$$

$$\text{or}\ a_{\overline{80|}}=42.22325$$

$$\therefore\ 2\ a_{\overline{80|}}=84.4465$$

$$\therefore\ 100\ v^{80}=23.99815$$

$$\text{or}\ v^{80}=.2399815$$

or using the bond table to obtain both equations we have

Investment Rate	Bond Rate	Unexpired term	Price
3.60%	4%	40 years	108.444650
3.60%	6%	40 years	150.667900

which means that at 1.8%

$$100\ v^{80}+2\ a_{\overline{80|}}=108.444650$$

$$100\ v^{80}+3\ a_{\overline{80|}}=150.667900$$

$$\therefore\ a_{\overline{80|}}=42.223250$$

as above.

INTEREST AND BOND VALUES.

CHAPTER IV.

ANNUITY BONDS.

As we saw in a previous chapter, a loan of \$10,000 at 5% compounded half yearly, may be repaid by ten semi-annual payments of \$10,000 $a_{\overline{10|}}^{-1}$ at 2½% = \$1,142.59 each.

At the end of the first half year the borrower will pay \$1,142.59, consisting of \$250 interest and \$892.59 which must be a return of capital, leaving \$9,107.41 capital still outstanding. At the end of twelve months the interest on this will be only \$227.69 so that the second payment of the annuity will contain \$914.90 capital. This process may be scheduled as follows:

Schedule illustrating the repayment of a loan of \$10,000. at 5% compounded half yearly by a five year semi-annual annuity of \$1,142.59 each half year:—

½ Yr	Semi-ann. payment of		Capital repaid to date	Capital still out-standing	½ Yr.
	Interest	Capital			
1	\$250.00	\$892.59	\$892.59	\$9,107.41	1
2	227.69	914.90	1,807.49	8,192.51	2
3	204.81	937.78	2,745.27	7,254.73	3
4	181.37	961.22	3,706.49	6,293.51	4
5	157.34	985.25	4,691.74	5,308.26	5
6	132.71	1,009.88	5,701.62	4,298.38	6
7	107.46	1,035.13	6,736.75	3,263.25	7
8	81.58	1,061.01	7,797.76	2,202.24	8
9	55.06	1,087.53	8,885.29	1,114.71	9
10	27.88	1,114.71	10,000.00	0.00	10

Bonds under which loans are repaid in this manner are known as Annuity Bonds. They are issued for a nominal amount and at a nominal rate of interest, but are of course bought as an investment to yield an appropriate investment rate of interest.

Annuity bond tables have been published and doubtless are used, but they are not satisfactory. Annuity bonds frequently provide for yearly payments, but the conventional yield basis is a rate compounded half yearly. It will be found more satisfactory to deal with such bonds by using the ordinary interest tables.

Consider a twenty year annuity bond for \$10,000 issued at a nominal 5% with half yearly payments and bought by an investor to yield him 5½% compounded half yearly.

The bond will be an obligation to pay \$10,000 $a_{\overline{40|}}^{-1}$ at 2½% = \$398.36 each half year for 20 years.

The purchaser will pay \$398.36 $a_{\overline{40|}}$ at 2¾% = \$9,592 to obtain the required yield.

The first payment of \$398.36 will consist of

\$263.78 being half a year's interest on the investment, and \$134.58 being a repayment of capital.

The capital then outstanding will be \$9,457.

The next payment of \$398.36 will consist of

\$260.07 being half a year's interest on capital outstanding, and \$138.29 being another repayment of capital.

The capital is thus written down from half year to half year until at the end of the 20 years it will be all written off.

Theoretically the investment might have been treated as one producing \$263.78 each half year for interest and \$134.59 as a sinking fund payment towards the reproduction of the capital at the end of the investment period. Such a sinking fund would at 5½% compounded half yearly, produce \$134.58 $s_{\overline{40|}}$ at 2¾% at the end of the twenty years. And \$134.58 $s_{\overline{40|}}$ at 2¾% = \$9,592. which is the amount of the capital invested.

But sinking funds can seldom be accumulated at the high rates yielded by such investments, so that problems such as the following must be solved.

A municipality issues a twenty year semi-annual annuity bond for $25,000, at a nominal 5%. What should a purchaser offer in order that he may secure 5¼% on his investment for the whole time and reproduce his capital intact by a sinking fund which he can accumulate at only 4%? All rates are to be compounded half yearly.

A reference to the interest table will shew, that the municipality must pay $25,000 $a^{-1}_{\overline{40|}}$ at 2½% = $995.91 each half year for twenty years.

But for every $10,000 invested by such a purchaser he should get each half year $262.50 for interest and also $10,000 $s^{-1}_{\overline{40|}}$ at 2% = $165.56 towards his sinking fund; or for each $428.06 in the semi-annual payment to be received, he can afford to pay $10,000. Therefore he will bid $\frac{995.91}{428.06}$ of $10,000 = $23,266.

To test this result:

The half yearly annuity payment is.............	$995.91
Half a year's interest at 5¼% on $23,266 is......	610.73
Therefore amount available for sinking fund is ...	$385.18

and $385.18 $s_{\overline{40|}}$ at 2% = $23,266, as it should.

To value an annuity bond with annual payments by the conventional half yearly compounded investment rate, we must find the equivalent half yearly payment at this rate.

Consider a 25-year annuity bond for $50,000 at 5% with yearly payments to be valued on a 4½% basis compounded half yearly.

The bond will produce $50,000 $a^{-1}_{\overline{25|}}$ at 5% = $3,547.62 at the end of each year for 25 years, and this is equivalent at the investment rate to $1,754.08 at the end of each half year, as may be seen from the table of equivalent payments.

Therefore the value required is $1,754.08 $a_{\overline{50|}}$ at 2¼%
= $52,331.92.

Had the bond been issued with quarterly payments on the same nominal basis, these quarterly payments would have been $50,000 $a^{-1}_{\overline{100|}}$ at 1¼% = $878.71 each, and the equivalent

half yearly payment at the investment rate would be $1,767.31, so that the value required would have been $1,767.31 $a_{\overline{50}|}$ at $2\frac{1}{4}\% = \$52,726.63$.

The repayment of a loan by an annuity which includes capital as well as interest may be shown algebraically as follows:

Schedule illustrating the repayment of a loan of $a_{\overline{n}|} = \dfrac{1-v^n}{i}$ by an annuity of 1 for n periods:—

Period r	The rth payment of 1 consists of		Total Capital repaid after the rth payment.	Capital still outstanding.
	Interest.	Capital.		
1	$1-v^n$	v^n	$v^n = a_{\overline{n}|} - a_{\overline{n-1}|}$	$a_{\overline{n-1}|}$
2	$1-v^{n-1}$	v^{n-1}	$v^n + v^{n-1} = a_{\overline{n}|} - a_{\overline{n-2}|}$	$a_{\overline{n-2}|}$
3	$1-v^{n-2}$	v^{n-2}	$v^n + v^{n-1} + v^{n-2} = a_{\overline{n}|} - a_{\overline{n-3}|}$	$a_{\overline{n-3}|}$
&c.	&c.	&c.	&c.	&c.
r	$1-v^{n-r+1}$	v^{n-r+1}	$a_{\overline{n}|} - a_{\overline{n-r}|}$	$a_{\overline{n-r}|}$
&c.	&c.	&c.	&c.	&c.
n	$1-v$	v	$v + v^2 + v^3 + \ldots + v^{n-1} + v^n = a_{\overline{n}|}$	0

It will be noticed that the repayments of capital form an increasing geometrical series with a common ratio $1+i$; or the capital contained in any annuity payment exceeds the capital contained in the previous payment by the interest on the latter capital, since the interest in any payment is less than that in the previous payment by the interest on the capital repaid by that previous payment.

When a loan of this character has been in progress for some years it may happen that the question of redemption will arise, and, in the absence of any pre-arranged and definite

agreement, disclose a wide divergence of views. Consider the following case.

Five years ago a loan of \$10,000 was made on a 5% basis to be repaid by a 15-year semi-annual annuity. The semi-annual payments which included capital as well as interest were \$10,000 $a_{\overline{30}|}^{-1}$ at $2\frac{1}{2}\%$=\$477.78 each. The tenth payment having just been made, the question of redemption has arisen. The borrower would certainly regard \$477.78 $a_{\overline{20}|}$ at $2\frac{1}{2}\%$ = \$7,448. as the capital outstanding, and this is the amount that would be shewn by a schedule such as that on page 59. The lender however might refuse to discount the future payments at more than, say, 4%, claiming that he could not reinvest the money with equivalent security to yield him more than that; and \$477.78 $a_{\overline{20}|}$ at 2%=\$7,812. which is considerably in excess of the borrower's idea of the capital outstanding.

Now suppose that the lender sells the security to a third party who buys it at a price to yield him $4\frac{1}{4}\%$ compounded half yearly and to allow for the redemption of his capital by a sinking fund which he can accumulate at only 4%. For every \$1000 in the purchase price, the semi-annual payment should contain for interest....................\$21.25
and for the sinking fund \$1000 $s_{\overline{20}|}^{-1}$ at 2%=\$41.16.

Therefore for each ....\$62.41 in the semi-annual payment such a purchaser would give \$1000.

He will therefore pay $\dfrac{477.78}{62.41}$ of \$1000=\$7,656. for the security and each payment of \$477.78 will give him \$162.69 for interest and \$315.09 towards his sinking fund.

Suppose further that after another two years and immediately after the fourteenth semi-annual payment, the question of redemption arises between the borrower and the present holder of his debt. The borrower will on the same basis as formerly estimate the capital outstanding to be \$477.78 $a_{\overline{16}|}$ at $2\frac{1}{2}\%$=\$6,237. The owner of the security will however properly regard it as being worth his purchase price less the accumulated amount of his sinking fund, that is

\$7,656.—\$315.09 $s_{\overline{4}|}$ at 2%

$=\$7,656-\$1,299=\$6,357.$

Indeed the holder may even refuse to discount the future payments due him at more than 4% at which rate he knows he can reinvest. On this basis he would claim $\$477.78\ a_{\overline{16}|}$ at 2% $=\$6,487$.

In the absence of any prior agreement, it would appear that if the borrower is anxious to repay, he can do so only on terms agreeable to the holder of his security. On the other hand, should the desire for redemption come from the holder of the security, he must accept the borrower's terms unless he can sell to better advantage some where else.

The annuity bond possesses obvious advantages from the issuer's point of view when the redemption of the capital must be effected by charges against income; but it is not a suitable investment, generally speaking, for private funds owing to the fact that the capital comes back in small sums unsuitable for reinvestment, even if the holder has the skill to separate each payment into its component parts.

In the hope of making the issue more attractive, some municipal corporations have adopted an ingenious plan by which the issuing corporation obtains all the advantages of an annuity bond, and yet each holder of any part of the security has a straight term bond with the usual coupons. Consider an issue of $50,000 of 5% bonds to run for 30 years.

The equivalent half-yearly payment is

$\$50,000\ a^{-1}_{\overline{60}|}$ at $2\frac{1}{2}\%=\$1,617.67$.

But the first half year's interest is only $1,250.00

Therefore the issue is divided as follows:—

One Bond for $367.67 due six months after issue and carrying one coupon for $9.19.

One Bond for $376.86 due 1 year after issue and carrying two coupons for $9.42 each.

One Bond for $386.28 due 1½ years after issue and carrying three coupons for $9.66 each.

One Bond for $395.94 due 2 years after issue and carrying four coupons for $9.90 each.

&c. &c. &c. &c.

One Bond for \$1,578.21 due 30 years after issue and carrying 60 coupons for \$39.46 each.

Under such an issue there will fall to be paid one bond and a diminishing number of coupons each year but the one bond and the coupons payable will always aggregate \$1,617.67 each half year.

One other unusual form of annuity bond may be illustrated.

An issue of \$20,000 at 5% with the usual half yearly coupons for the first five years; after 5 years the loan to be repaid by a semi-annual annuity running for twenty years longer.

Such a bond will produce \$500 each half year for 5 years, to be followed by semi-annual payments of \$20,000 $a_{\overline{40}|}^{-1}$ at $2\frac{1}{2}\%$ = \$796.72 each for the next twenty years.

The value of such an issue on a $5\frac{1}{4}\%$ basis compounded half yearly is obviously

\$500 $a_{\overline{10}|}$ at $2\frac{5}{8}\%$	=	\$ 4,347.90
+\$796.72 $(a_{\overline{50}|} - a_{\overline{10}|})$ at $2\frac{5}{8}\%$	=	15,114.71
		\$19,462.61

Annuity bond tables giving prices on a percentage basis are, as has been said, not at all satisfactory to the man who fully understands the assumptions underlying the given prices; and to one who does not clearly appreciate these assumptions such bond tables are dangerous tools to use. It should be noted however that the ordinary straight term bond tables enable one to value any annuity bonds, since from such tables the interest functions v^n and $a_{\overline{n}|}$ may be readily deduced as indicated on page 58.

When an annuity bond is bought between two payment dates on a yield basis the interest adjustment may be made in one of three ways, all regardless of the nominal bond rate.

Consider an annuity bond, under which 20 annual payments of \$1000. each remain to be made, bought 3 months after a payment to yield the purchaser 4% compounded half yearly.

We should first ascertain the equivalent half yearly payment at 4%—this will be \$495.05.

First Method:

The value as at last payment date

$= \$495.05\ a_{\overline{40}|}$ at 2%		= \$13,542.33
add 3 months' interest at 4% on \$13,542.33 =	\$135.42	
less 3 months' discount at 4% on \$135.42 =	1.35	134.07
Price		\$13,676.40

Second Method:

The value as at last payment date

$= \$495.05\ a_{\overline{40}|}$ at 2%	\$13,542.33
Add 3 months' interest at 4% on \$13,542.33	135.42
Price	\$13,677.75

Third Method:

The value as at next payment date $= \$495.05\ (1 + a_{\overline{39}|})$ at 2%	= \$13,813.18
One dollar at interest to next payment date will amount to	\$1.01
Therefore price is \$13,813.18 ÷ 1.01	= \$13,676.41

The true price is $\$13{,}542.33 \times (1.02)^{\frac{1}{2}}$

$= \$13{,}542.33 \times 1.009955$

$= \$13{,}677.14$

The error of the first method is 74 cents in defect.

The error of the second method is 61 cents in excess.

The error of the third method is 73 cents in defect.

The first method is the one generally used and, as with straight term bonds, it will always give an error in defect which will always be about half the "discount" item.

INTEREST AND BOND VALUES.

CHAPTER V.

Bonds from the Issuer's Point of View.

We have been looking at bond issues from the investor's point of view. It may not be out of place to consider such issues from the point of view of the issuing corporation.

As to the nominal bond rate, it is within limits immaterial theoretically what bond rate is used. If the corporation can borrow money at 4% compounded half yearly, then a 4% bond can be issued at par. A 3½% bond will sell below par and a 4½% bond will sell above par. The following prices will be found in any table of bond values.

A 3½% 20-year bond to yield 4% sells at 93.16.

A 4 % 20-year bond to yield 4% sells at 100.00.

A 4½% 20-year bond to yield 4% sells at 106.84.

The difference 6.84 in either case is merely the cash value of the difference in the coupons. The 3½% bond will produce less cash and entail a correspondingly smaller annual charge for coupons. The 4½% bond will produce more cash and entail a correspondingly larger annual charge for coupons. The corporation cannot borrow more cheaply by printing a smaller bond rate in the securities it is about to issue. It will generally be advisable, however, to insert a bond rate as near as may be to the investment rate at which the securities will sell, since in many cases investors find it inconvenient to buy at either a considerable premium or a considerable discount. Bonds that are selling to the ultimate public at a discount "look cheap" and may prove attractive on that score to some buyers; but, as most bond-salesmen know, many private investors do not like bonds quoted at a discount, fearing that such a quotation implies a weakness in the security; while others compare the coupons with the price regardless of the fact that the coupons last for only a limited time and that the bond will

ultimately be paid off at par. To a man of the latter type the $4\frac{1}{2}\%$ bond selling at 106.84 seems to give a better return than the $3\frac{1}{2}\%$ bond selling at 93.16.

It is worth noticing that if the issuing corporation must meet the bonds at maturity by means of a sinking fund accumulating at a rate lower than that at which they can sell the issue, then the bond issued at a premium will be the cheaper proceeding. Suppose that in the issue we are considering it is only possible to accumulate the sinking fund at 3% compounded half yearly. Then $\$1000\ s^{-1}_{\overline{40}|}$ at $1\frac{1}{2}\% = \$18.43$ will be required for the sinking fund to retire each $1000. of the issue no matter what the bond rate may be.

The $3\frac{1}{2}\%$ bond will entail a semi-annual charge of $35.93 per $1000, for coupons and sinking fund and will produce $931.60 in cash.

The 4% bond will entail a semi-annual charge of $38.43 in return for $1000.00 cash.

The $4\frac{1}{2}\%$ bond will entail a semi-annual charge of $40.93 in return for $1068.40.

$$\text{But} \qquad \frac{35.93}{931.60} = .03857$$

$$\frac{38.43}{1000.00} = .03843$$

$$\text{and} \qquad \frac{40.93}{1{,}068.40} = .03831.$$

This shews that the $4\frac{1}{2}\%$ bond is really the cheapest way to borrow.

As to the term for which the bonds should run it is obviously generally better to issue long term bonds when investment rates are low and *vice versa*. The standard illustration usually referred to is the action of the British Government in selling 3% annuities during the struggle against Napoleon at apparently reckless discounts, instead of putting out short term bonds to be redeemed as the public credit rose by refunding issues at better prices. The exchequer bonds issued during the Boer War and the numerous recent

short term notes of American railways are cases in point Similarly a small town in a rapidly developing district of the North West may in a few years be in a much stronger position financially and might be well advised to issue bonds for only a moderate term in spite of a general low level of investment rate that may be ruling at the time.

For many years it was commonly believed that the general trend of the interest rate from high class securities was to be continuously and indefinitely downwards; but that trend has been reversed during the immediate past few years and the recent serious rise in the price level of commodities or fall in the purchasing power of money has induced a belief that lenders will be forced to protect themselves by demanding higher investment rates of interest. Such speculations regarding the future of the rate of interest are however quite beyond our scope.

The choice between annuity bonds or serial bonds and the popular straight term issue, must frequently present itself for decision by the issuing corporation. The straight term bond undoubtedly commands a broader market and is often supposed to indicate a certain financial prestige, so that municipal issues for example from the larger centres are usually in this form, while the annuity bond is generally adopted by the smaller municipalities.

Of course if the security to be pledged should be of a depreciating nature, such as railway rolling stock, it may well be that the issuing corporation has no choice—that only a serial issue or annuity bonds would be acceptable to the purchaser. But such questions lie outside our range. Assuming that the test of figures is the only test, we can always decide the point.

Consider the following case. It is intended to issue \$10,000 of 20-year 5% bonds and it has been ascertained that if the issue be made in the form of straight term bonds they can be sold on a yield basis of 4⅞%; but if the issue is to be in the form of annuity bonds they can only be sold on a yield basis of 5¼%. Assuming that the straight term issue will demand a sinking fund that must be accumulated at 4%, which will be the cheaper form of the issue ?

We will assume that all interest rates are as usual to be compounded half yearly.

The annuity bonds will entail a semi-annual charge of $\$10,000\ a^{-1}_{\overline{40|}}$ at $2\frac{1}{2}\% = \$398.36$.

The straight term issue will demand each half year

For the coupons	$250.00
For the sinking fund, $\$10,000\ s^{-1}_{\overline{40|}}$ at 2%	165.56
	$415.56

The annuity bonds will sell for $\$398.36\ a_{\overline{40|}}$ at $2\frac{5}{8}\% = \$9,793$.

The straight term bonds will sell for the price quoted in a bond table, $10,159.

$$\text{But} \qquad \frac{398.36}{9,793.} = .04068$$

$$\text{and} \qquad \frac{415.56}{10,159.} = .04091.$$

So that the annuity bond will be really the cheaper method of borrowing in this case.

But if the sinking fund can be invested in the bonds it is meant to redeem or in other securities of the issuing corporation there should be no inducement to put out an annuity bond.

The treatment of the bond issue on the books of the issuing corporation is hardly within our scope; but it may be well to point out that any premium or discount at which the bonds may be sold should never find its way into a revenue account. If an issue of $50,000 is sold at 97, the initial outstanding debt in respect of this issue is $48,500 not $50,000; and each coupon as it is met will not pay the full interest on the debt. The difference between the full interest payable and the coupon will increase the debt at each coupon date, until at the due date of the bond the amount will be increased to the $50,000 then payable. On the other hand if the bonds be sold at 104, the initial debt is $52,000 and each coupon will contain some repayment of that debt in addition to the full interest payable upon it, until at the due date of the bond the debt has been diminished by $2,000 leaving only the $50,000 then payable.

INTEREST AND BOND VALUES.

CHAPTER VI.

SOME PROBLEMS.

1. Twenty thousand dollars was deposited with a Trust Company 30 years ago. Each half year, during the past 25 years, \$500 has been drawn out, the last payment of \$500 having just been made. How much should now be to the credit of the account, assuming 4% compounded half yearly?

Had nothing been drawn out, there would have been \$20,000 $(1+i)^{60}$ at 2%	\$65,620.62
But the half yearly drawings now amount to \$500 $s_{\overline{50}|}$ at 2%	\$42,289.70
leaving a balance of	\$23,330.92

Or, five years after the original deposit it would have grown by interest to \$20,000 $(1+i)^{10}$ at 2%=\$24,379.89. Half a year's interest on this at 4% is =\$487.60.

By drawing out \$500, the interest was overdrawn by \$12.40. These excess payments would in 25 years amount to \$12.40 $s_{\overline{50}|}$ at 2%=\$1,048.78. Deducting this from the \$24,379.89 would leave \$23,331.11.

This result is 19 cents in excess of the more accurate result above. The error is due to the fact that amounts are taken to the nearest cent, and so may in any case contain an error not greater than a half cent. The interest item of \$487.60 should strictly be \$487.5978.

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2. A is lending \$10,000 to B for 5 years at 5% and B wishes to have the right of redemption at the end of any year during the currency of the loan at a pre-arranged price. Draw up a schedule shewing the redemption price at the end of each year so as to secure to A the full 5% on the whole loan for the whole

time though he can reinvest at only 3%. Interest is payable half yearly.

At the end of the first year after the interest has been paid the redemption price should be

$10,000+$100 $a_{\overline{8}\|}$ at 1½%	$10,748.59
At the end of the second year $10,000 + $100 $a_{\overline{6}\|}$ =	10,569.72
At the end of the third year $10,000+$100 $a_{\overline{4}\|}$. . =	10,385.44
At the end of the fourth year $10,000+$100 $a_{\overline{2}\|}$. =	10,195.59

To check the fourth year redemption price.

A has had his interest up to date. He now receives and deposits at 3% the sum of	$10,195.59
Add six months' interest at 3%	152.93
	$10,348.52
Deduct A's half yearly interest agreed upon. .	250.00
	$10,098.52
Add six months' interest at 3%	151.48
	$10,250.00
Deduct A's half yearly interest agreed upon	250.00
	$10,000.00

---

3. Five years ago two loans of $10,000 each were granted at 4%, the one repayable by an annuity for twenty years to include principal and interest and the other by equal instalments of $500 a year with interest on the principal outstanding. A third party has agreed to take over both loans on a 3½% basis. What should he pay the original lender? Interest is to be compounded yearly.

Under the first loan there are 15 yearly payments of $10,000 $a^{-1}_{\overline{20}|}$ at 4% = $735.82 each yet to be made

The value of these on a 3½% basis is

$735.82 $a_{\overline{15}|}$ at 3½% = $8,474.74.

Under the second loan there are 15 yearly payments of

principal, \$500 each, with interest at 4% on the outstanding principal from time to time.

Makeham's formula is here most suitable.

$C = \$7{,}500.00$

$K = \$500\, a_{\overline{15|}}$ at 3½% .... $= \$5{,}758.71$

$C - K = \$1{,}741.29$

$\frac{j}{i} = \frac{4}{3\frac{1}{2}} = \frac{8}{7} \quad \therefore \frac{j}{i}(C-K)$ .... $= 1{,}990.05$

The value being.......................... \$7,748.76

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4. A 6% bond with half yearly coupons is due on 1 July 1925 and will be redeemed at 105. What will the bond yield if bought at 112½ on 1 May 1911 by a purchaser who will set up a sinking fund at 3% compounded half yearly to replace the premium at which he bought?

Taking 1 point to roughly represent interest on the investment for the 2 months before the next coupon date. The capital invested as at 1 July 1911 is 113½ − 3 = 110.5.

To replace the 5½ points by which this exceeds the redemption price the sinking fund will demand $5.5\, s^{-1}_{\overline{28|}}$ at 1½% = .1595 out of every coupon, leaving 2.8405 for half a year's interest on an investment of 110.5 or 2.57% i.e. 5.14% per annum.

Our initial rough assumption of 1 as 2 months' interes ton 112½ was at the rate of 5⅓%.

To test the accuracy of our result—5.14%

The price as at 1 May 1911 was..................	112.500
Add 2 months' interest at 5.14%.................	.964
	113.464
Deduct coupon due 1 July......................	3.
Capital invested as at 1 July 1911............	110.464

Sinking fund requirement $= 5.464\, s^{-1}_{\overline{28|}}$ at 1½%

$= .1585$

Therefore the investment produces 2.8415 each half year on a capital of 110.464 which will remain intact.

But this is at the rate of $\frac{5.683}{1.10464} = 5.14\%$.

---

5. The debentures of a company which bear interest at 5% payable half yearly are redeemable in 10 years time at 110, and are quoted at a price which yields 5% compounded half yearly. It is proposed to change these debentures into a perpetual 4½% debenture stock with interest payable half yearly. How much of the new debenture stock should be given for each \$100 of the old issue ?

The price of the present debentures must be

$110\ v^{20} + 2.5\ a_{\overline{20|}}$ at $2\frac{1}{2}\%$

$= 67.130 + 38.973 = 106.103.$

If the new debenture stock will sell on the same yield basis its price will be 90.

Therefore each \$100 of the old debentures should be worth $\$\frac{10{,}610.3}{90.} = \$117.89$ of the new stock, and the change might be effected at the rate of 118 of the new for 100 of the old.

---

6. A bond redeemable in 12 years at par and bearing interest at 5% payable half yearly is bought at 105. Find the yield to the purchaser who sets up a sinking fund at 3% compounded half yearly to replace the decrease in capital.

The sinking fund will demand $5\ s^{-1}_{\overline{24|}}$ at $1\frac{1}{2}\% = .17462$ out of each coupon, leaving 2.32538 for a half year's interest on a capital of 105 which will remain intact. That is 4.429%.

---

7. A 5% debenture due 8 years hence and carrying half yearly coupons for interest is quoted in the market at 109. It is proposed to convert these debentures into 4½% debentures of the same amount with the same security. When should the new issue be made payable so as not to disturb the market price ?

Reference to a bond table shews that the quoted price corresponds to a yield of 3.69%.

At this yield a 4½% debenture should have 14½ years to run.

---

8. A Building Society grants loans repayable by 20 equal half yearly instalments including principal and interest, upon the basis of charging 7% interest on the total sum advanced and allowing 3% interest on sums repaid. The company advances two-thirds of the value of his property to a customer, but borrows half the value by pledging its mortgage at 5%. What rate of interest does the Company make on such transactions?

Consider a property worth $3,000.

The company will lend $2,000 and borrow $1,500.

The company's customer must pay each half year

$70.00 interest and also

86.49 = $2,000 $s^{-1}_{\overline{20}|}$ at 1½%, towards redemption.

Or $156.49 in all each half year.

The company must pay $37.50 each half year for interest. At the end of the 10 years the customer's balance in the redemption account will wipe out his loan.

Regarding this as a single transaction and assuming that the 3% on sums repaid by the customer can be obtained from the Bank, the company gets $32.50 each half year for an investment of $500. which is 13%. Regarding this as one of a number of similar transactions and assuming that the company can reinvest the sums repaid by its customers in other loans of the same character, the rate of interest made is that at which $(156.49 - 37.50)\, a_{\overline{20}|} = 500 + 1500v^{20}$, which is about $22\frac{1}{8}$%, or 44¼% per annum.

---

9. The value of an annuity for 20 years of which the payments are successively 20, 19, 18, etc. is 150. What rate of interest does this return?

Here $150 = 20v + 19v^2 + 18v^3 + \ldots + 3v^{18} + 2v^{19} + v^{20}$

$\therefore (1+i)\,150 = 20 + 19v + 18v^2 + \ldots + 3v^{17} + 2v^{18} + v^{19}$

$\therefore 150\,i = 20 - v - v^2 - v^3 - \ldots - v^{19} - v^{20}$

$= 20 - a_{\overline{20}|}$.

The required rate is that at which $\dfrac{20 - a_{\overline{20}|}}{i} = 150.$

$$\text{At } 5\%, \quad \frac{20 - a_{\overline{20}|}}{i} = \frac{7.5378}{.05} = 150.76.$$

$$\text{At } 5\tfrac{1}{8}\%, \quad \frac{20 - a_{\overline{20}|}}{i} = \frac{7.6688}{.05125} = 149.64.$$

The rate required is about $(5 + \frac{76}{112} \text{ of } \frac{1}{8})\%$

about $5\frac{1}{12}\%$.

---

10. A certain stock of the nominal value of $100 bears a semi-annual dividend of 30 cents and a quinquennial bonus of $3.00. Thirty months after the last payment of the bonus and immediately after a semi-annual dividend has been paid the stock is quoted at 23. Find the investment yield at this price.

At a yield of i per half year, the quinquennial bonuses of $3 each may be turned into half yearly bonuses of $\dfrac{v^5}{a_{\overline{10}|}}$ of $3 each, the first one to come in six months hence.

Therefore 23 is the value of a perpetuity of $.30 + \dfrac{v^5}{a_{\overline{10}|}}$ of 3.

$$\text{Or } 23\,i = .30 + \frac{3v^5}{a_{\overline{10}|}}$$

$$\therefore\ 23 = \frac{.30}{i} + \frac{3v^5}{1 - v^{10}}.$$

To get a rough approximation to i we may regard the quinquennial bonuses as being worth 30 cents each half year, giving $23i = .60$ or $i = 2.6\%$ roughly.

Try $2\frac{5}{8}\%$. At this rate the price would be

$$\frac{.30}{.02625} + \frac{2.63544}{.22826} = 11.429 + 11.546 = 22.975.$$

Try $2\frac{1}{2}\%$. At this rate the price would be

$$\frac{.30}{.025} + \frac{2.65155}{.2188} = 12.000 + 12.119 = 24.119.$$

At 2.625% the price would be 22.975.
At 2.500% the price would be 24.119.
Therefore the true yield is $(2.625-x)\%$
where $x : .125 :: .025 : 1.144$

$$\text{or } x = \frac{.003125}{1.144} = .0027$$

and the yield required is 2.6223 each half year or 5.245% per annum.

---

11. A government loan of \$1,000,000 bearing interest at 5½% payable half yearly is to be redeemed at 110 percent by a half yearly annuity of fixed amount including principal and interest extending over 10 years. What is the amount of the fixed half yearly payment?

Regarding the loan as one of \$1,100,000 at 5% to be redeemed at par by a half yearly annuity of fixed amount in 10 years, the half yearly payment should be
$\$1,100,000\ a^{-1}_{\overline{20}|}$ at $2\frac{1}{2}\% = \$70,561.84$.

12. An issue of \$1,000,000 of 4½% bonds with half yearly coupons is redeemable at 105 by annual drawings spread equally over 5 years; the first drawing to take place 3 years after issue. Find the issue price to yield 5% compounded half yearly.

Using Makeham's formula, we have

$C = \$1,050,000$

$K = \$210,000\ (v^6+v^8+v^{10}+v^{12}+v^{14})$ at $2\frac{1}{2}\%$

$= \dfrac{100}{202.5}$ of $\$210,000\ (v^5+v^6+v^7 \ldots +v^{13}+v^{14})$

$= \$103,703.70\ (a_{\overline{14}|} - a_{\overline{4}|}) \ldots\ldots\ldots\ldots\ldots = \$822,260.50$

Now $j = 4\frac{1}{2}$ on 105 or $4\frac{2}{7}\%$

$\therefore\ \dfrac{j}{i}(C-K) = \dfrac{4\frac{2}{7}}{5}$ of $\$227,739.50 \ldots\ldots = 195,205.29$

$\$1,017,465.79$

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13. A loan of \$100,000 is to be paid off in 30 years by quinquennial instalments, the first of which is to be made at the

end of 5 years. The loan bears interest at 4% payable half yearly and the semi-annual sum set aside for the service of the loan is \$2,912.25. Find the rate of interest at which the sinking fund should accumulate by half yearly compounding during each quinquennium.

The semi-annual coupons call for \$2,000. leaving \$912.25 for the sinking fund. Each fifth year the sinking fund will be invested in the loan itself, so that the coupons on the bonds purchased for the sinking fund will increase the half yearly contribution to that fund. For each unit in \$912.25 the sinking fund will grow as follows:—

End of 5th year:—$s_{\overline{10}|}$ at rate i, the unknown rate.

End of 10th year:—$s_{\overline{10}|}+(1+j\, s_{\overline{10}|})\, s_{\overline{10}|}$ where $j=.02$, the bond rate.

End of 15th year:—$s_{\overline{10}|}\{1+(1+j\, s_{\overline{10}|})+(1+j\, s_{\overline{10}|})^2\}$

and so on.

End of 30th year:—$s_{\overline{10}|}\{1+(1+j\, s_{\overline{10}|})+(1+j\, s_{\overline{10}|})^2+ \dots +(1+j\, s_{\overline{10}|})^5\}$

$$= s_{\overline{10}|}\frac{(1+j\, s_{\overline{10}|})^6-1}{1+j\, s_{\overline{10}|}-1} = \frac{1}{j}\{(1+j\, s_{\overline{10}|})^6-1\}$$

$$\therefore 912.25 \times \{(1+j\, s_{\overline{10}|})^6-1\} = j \times 100{,}000 = 2{,}000.$$

$$\therefore (1+j\, s_{\overline{10}|})^6 = \frac{2{,}912.25}{912.25}$$

$$\text{or, } 6 \log (1+j\, s_{\overline{10}|}) = \log 2{,}912.25 - \log 912.25$$

$$= .5041148$$

$$\log (1+j\, s_{\overline{10}|}) = .0840191$$

$$\therefore 1+.02 \times s_{\overline{10}|} = 1.21344.$$

The required rate is therefore that at which $s_{\overline{10}|} = 10.672$. Reference to the tables shows this to be $1\frac{7}{16}$% per half year or $2\frac{7}{8}$% per annum compounded half yearly.

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14. A financial house has agreed to underwrite a foreign government loan of \$1,000,000 at 97. The loan will bear interest at the rate of 4% per annum payable yearly. It is under discussion whether repayment should be made by means of an

accumulative sinking fund of 2% or by uniform annual drawings of \$20,000. What difference would there be in the rates of interest paid over the whole transaction by these methods?

If the accumulative sinking fund is adopted the loan will be redeemed in n years where $100 = 6\ a_{\overline{n}|}$ at 4% i.e. in 28 years, and the rate of interest paid is that rate at which $97 = 6\ a_{\overline{28}|}$ or $a_{\overline{28}|} = 16.167$

At $4\frac{1}{4}\%$, $a_{\overline{28}|} = 16.193$.

At $4\frac{3}{8}\%$, $a_{\overline{28}|} = 15.966$.

$\therefore$ a close approximation to the desired rate will be

$$4\tfrac{1}{4} + \tfrac{26}{227} \text{ of } \tfrac{1}{8} = 4.264\%.$$

If redemption is to be made by uniform annual drawings we can approximate to the rate by using Makeham's formula

$$i = j + \frac{C - A}{C - K}\, i \quad \text{(see page 55)}$$

We may try $4\frac{1}{8}\%$ as our rough guess.

At this rate $K = 2\ a_{\overline{50}|} = 42.06$

$C - A = 100 - 97 = 3$: and $C - K = 57.94$

$$\therefore \quad i = .04 + \frac{3}{57.94} \times .04125$$

$$= .04 + .00214 = .04214 \text{ or } 4.214\%.$$

Our guess was too low: try $4\frac{1}{4}\%$.

At this rate $K = 2\ a_{\overline{50}|} = 41.19$.

$C - K = 58.81$

$$\therefore i = .04 + \frac{3}{58.81} \times .0425$$

$$= .04 + .00217 = .04217 \text{ or } 4.217\%$$

A trial rate of $4\frac{1}{8}\%$ gave 4.214%.

A trial rate of $4\frac{1}{4}\%$ gave 4.217%.

$\therefore$ The true rate is $(4\frac{1}{8} + x)\%$,

where $x : .125 :: x - .089 : .003$

or $.003\,x = .125x - .011$

$$\therefore x = \frac{.011}{.122} = .090$$

And the rate required is $4.125+.090=4.215\%$.

By the former method the government would be paying 4.264% for its money and by the latter 4.215%, a difference of $.049\%$.

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15. An issue of bonds of the nominal value of \$100,000 is made on 1 April 1911. The bonds bear interest at 4% per annum, payable half yearly, and are redeemable by drawings as follows:

\$10,000 on 1 April 1912 at 101.
\$10,000 on 1 April 1913 at 102.
&c. &c. &c.
\$10,000 on 1 April 1921 at 110.

What should be the issue price to yield 5% compounded half yearly?

The price of the bonds $=A_1+A_2$

where $A_1=$ price disregarding the premiums on redemption and $A_2=$ cash value of the premiums payable on redemption. Makeham's formula is appropriate to find A_1.

$$A_1=K+\frac{j}{i}(C-K)$$

where $C=100{,}000$

$K=$ present value of 10 annual payments of \$10,000 each

$=$ present value of 20 semi-annual payments of \$4,938.30 each (see table, page 30)

$=\$4{,}938.30\ a_{\overline{20}|}$ at $2\frac{1}{2}\%$ . . . . $=\$76{,}984$

$\frac{j}{i}(C-K)=\frac{4}{5}$ of \$23,016 . . . . $=\ \ 18{,}413$

$\therefore A_1$ . . . . $=\$95{,}397$

$$A_2=100\ (v^2+2v^4+3v^6+\ldots+9v^{18}+10v^{20}) \text{ at } 2\tfrac{1}{2}\%$$
$$v^2A_2=100\ (v^4+2v^6+3v^8+\ldots+9v^{20}+10v^{22})$$
$$\therefore (1-v^2)A_2=100\ (v^2+v^4+v^6+\ldots+v^{20})-1000\ v^{22}$$
$$=49.383\ a_{\overline{20}|}-1000\ v^{22}$$

$$\therefore A_2 = \frac{769.84-580.86}{.048186} = \$3,922$$

so that the full price $A_1+A_2=\$99,319$.

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16. A loan of \$5,000. was made 5 years ago to be repaid in 25 years by equal half yearly instalments of \$100. each with interest at the rate of 4½% per annum on the capital outstanding in each half year. Just after the 10th half yearly payment the lender offers his security for sale. How much should be given by a purchaser so that he may realise 5%, payable half yearly on his investment while replacing his capital at 4% compounded half yearly?

There is now \$4,000 of the loan outstanding.

The next half yearly payment will be \$100 capital and \$90 interest.

The following half yearly payment will be \$100 capital and \$87.75 interest.

The following half yearly payment will be \$100 capital and \$85.50 interest.

Or the future payments of interest will decrease by \$2.25 each half year.

If we denote the payments made by the borrower as

$x,\ x-y,\ x-2y,\ x-3y, \ldots x-39y$

where $x=\$190.00$ and $y=\$2.25$, and also denote the price paid by the purchaser as P, and write j for .025,

then we have available for the sinking fund

$x-jP,\ x-y-jP,\ x-2y-jP$, &c. in successive half years, so that

$$(x-jP)(1.02)^{39}+(x-y-jP)(1.02)^{38}+\ldots+(x-39y-jP)=P$$

or $$(x-jP)(1.02)^{40}+(x-y-jP)(1.02)^{39}+\ldots$$
$$\ldots+(x-39\,y-jP)(1.02)=P\,(1.02)$$

$$\therefore (x-jP)(1.02)^{40}-y\,(1.02)^{39}-y\,(1.02)^{38}\ldots\ldots-y\,(1.02)$$
$$-(x-39y-jP)=P\times .02$$

$$\therefore (x-jP)(1.02)^{40}-y\,s_{\overline{40}|}\,2\%-x+40y+jP=P\times .02$$

$$\therefore P=\frac{x\,(1.02)^{40}-y\,s_{\overline{40}|}\,2\%-x+40y}{j\,(1.02)^{40}+.02-j}$$

$$= \frac{190 \times 2.20804 - 2.25 \times 60.40198 - 190 + 90}{.025 \times 2.20804 - .005}$$

$$= \frac{183.624}{.050201} = \$3,657.80.$$

---

17. A loan of \$18,323. is to be repaid in five years by an annuity calculated at 3% to include principal and interest at 5%. If the first payment of the annuity is \$4,120.26 and the payments are to increase in geometrical progression, prove that their common ratio is 1.03 and draw up a schedule shewing the repayment of the loan.

The annual interest demanded is \$916.15.

If the successive payments be P, Pr, Pr^2, Pr^3, Pr^4, we have

$$(P-916.15)\,(1.03)^4 + (Pr-916.15)\,(1.03)^3$$
$$+ (Pr^2-916.15)\,(1.03)^2 + (Pr^3-916.15)\,(1.03)$$
$$+ (Pr^4-916.15) = 18,323.$$

Or $P\,(1.03)^4 + Pr\,(1.03)^3 + Pr^2\,(1.03)^2 + Pr^3\,(1.03) + Pr^4$

$= 916.15\ s_{\overline{5}|}$ at $3\% + 18.323$

$= 4,864 + 18,323 = 23,187$

But $P = 4,120.26$

And $\frac{23,187}{4,120.26} = 5.6275 = 5 \times 1.1255 = 5\,(1.03)^4$

$\therefore r = 1.03$

The schedule illustrating the repayment is as follows:

Year.	Annuity Payment.	3% on Capital repaid. This comes from re-investm't.	The Annuity payment contains: 5% on Capital outstanding.	2% on Capital repaid.	A Capital Repayment.	Capital repaid to date.	Capital still outstanding.
1	\$4,120.26	\$ 0.00	\$916.15	\$ 0.00	\$3,204.11	\$ 3,204.11	\$15,118.89
2	4,243.87	96.12	755.95	64.08	3,423.84	6,627.95	11,695.05
3	4,371.19	198.84	584.75	132.56	3,653.88	10,281.83	8,041.17
4	4,502.33	308.45	402.06	205.64	3,894.63	14,176.46	4,146.54
5	4,637.40	425.29	207.33	283.53	4,146.54	18,323.00	0.00
		\$916.15					

18. A Timber berth of 30,000 acres of pine estimated to produce 120 million feet of lumber is bought for \$800,000. and a ground rent of \$8 per square mile. Logging may begin after 3 years at the rate of 6 million feet per annum, and at a cost of \$8 per thousand feet to the mill. The stumpage dues are \$3 per thousand. The mill charges \$3.50 per thousand. At what price should the lumber be sold in order to make a net profit of 10% per annum on the proprietor's outlay? It is to be assumed that lumber prices will rise 2% per annum; also that \$500,000 can be raised at once by the issue of 5% bonds at par, the bonds to be redeemed at the rate of \$50,000 a year, at the ends of the years sixth to fifteenth after issue.

Assuming that each year's cut is got out, put through the mill and marketed during 12 months, the final proceeds will come in 23 years after the purchase. At that time the accumulations of the outgo at the yield rate will amount to

Original outlay, \$300,000 $(1+i)^{23}$	=	\$2,686,290
Ground rent, \$375 $s_{\overline{23}\rceil}$	=	29,829
Stumpage, logging and mill costs, \$87,000 $s_{\overline{20}\rceil}$.	=	4,982,925
		\$7,699,044

plus the payments for interest and redemption of bonds as follows:

For coupons during the first six years,
\$25,000 $s_{\overline{6}\rceil}(1+i)^{17}$ \$974,957

For capital redemption from the end of the 6th to the end of the 15th year inclusive,
\$50,000 $s_{\overline{10}\rceil}(1+i)^{8}$ 1,708,165

For coupons from the end of the 7th to the end of the 15th year inclusive:

$$\$22{,}500\ (1+i)^{16}+20{,}000\ (1+i)^{15}+\ldots\ +2{,}500\ (1+i)^{8}$$

$$=\$2{,}500\ \{9\ (1+i)^{16}+8\ (1+i)^{15}+7\ (1+i)^{14}+\ \ldots\ +2\ (1+i)^{9}+(1+i)^{8}\}$$

$$=\$2{,}500\ \frac{10\ (1+i)^{17}-(s_{\overline{18}\rceil}-s_{\overline{8}\rceil})}{i}\ \ldots\ldots\ldots\ldots \quad 819{,}075$$

\$3,502,197

The total outgo for all purposes at the yield rate will amount to $11,201,241.

If $\$p$ per thousand is the first price realised, the accumulated income on the same basis is

$$\$6{,}000\left\{p(1+i)^{19}+p(1.02)\ (1+i)^{18}+\ldots\ldots p(1.02)^{18}(1+i)+ +p(1.02)^{19}\right\}$$

$$=\$6{,}000\ \frac{(1+i)^{20}-(1.02)^{20}}{(1+i)-(1.02)}\ p,\ \text{ but } i=.10$$

$$=\$6{,}000\ \frac{5.24155}{.08}\ p=\$393{,}116\ p$$

$$\therefore\ p=\frac{11{,}201{,}241.}{393{,}116}=\ 28.49$$

The first year's cut should sell for $28.50 per M.

The second year's cut should sell for $29.10 per M.

The last year's cut should sell for $41.50 per M.

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19. A provincial government makes an issue of 3½% bonds to obtain the money necessary to replant an area of 12,500 acres with white pine. The initial expenses together with the purchase price of the property amount to $75,000. During each of the first five years $1000 will be spent in road making. Four per cent of the area must be set aside for roads, fire lines &c. Planting is done at the rate of 1000 acres a year at $10 an acre. Salaries and expenses amount to $5,000 a year. After 30 years, 50 cents per acre can be secured on each area from thinnings. This can be repeated at the end of each 10 year period and the proceeds will increase at the rate of 2% per annum. At the end of 60 years the final harvest will produce $300 per acre.

Assuming that bonds will be sold at par as the money may be needed, what will be the maximum issue outstanding? When will this maximum occur? When can the bonds be retired? What will the forest be worth when the last of the issue has been paid off?

The total area is........12,500 acres.
Road allowance........ 500

Leaving...............12,000 acres available for planting at the rate of 1000 acres a year for 12 years, and costing \$10,000 each year.

No income can come from the forest until the end of the 30th year when the proceeds from thinnings will begin and continue as follows:—

End of 30th year—\$500 a year for 12 years.
End of 40th year—\$500 $(1.02)^{10}$ a year for 12 years.
End of 50th year—\$500 $(1.02)^{20}$ a year for 12 years.

At the end of the 60th year the final harvest will produce \$300,000 a year for 12 years.

The income from thinnings will never be enough to pay interest on the bonds outstanding at the time.

At the end of the 60th year the debt on the property will amount to:—

Purchase price at $3\frac{1}{2}\%$ interest for 60 years, \$75,000 $(1+i)^{60}$..........................	=	590,857
Cost of road making, \$1,000 $s_{\overline{5}|}\,(1+i)^{55}$	=	35,570
Cost of planting, \$10,000 $s_{\overline{12}|}\,(1+i)^{48}$........	=	761,286
Cost of maintenance, \$5000 $s_{\overline{60}|}$............	=	982,584
		\$2,370,297
less the proceeds from sales of thinnings, \$500 $\left\{ s_{\overline{12}|}\,(1+i)^{18} + (1.02)^{10}\, s_{\overline{12}|}\,(1+i)^{8} + (1.02)^{20}\, s_{\overline{10}|} \right\}$....................	=	33,997
		\$2,336,300

And this will be the maximum issue outstanding. The harvest from the first 1000 acres will now produce \$300,000 leaving \$2,036,300 outstanding.

The whole issue will be redeemed in n years more where

$$\$300{,}000\, s_{\overline{n}|} + \$500\ (1.02)^{20}\ s_{\overline{2}|}\,(1+i)^{n-2} - \$5{,}000\, s_{\overline{n}|} = \$2{,}036{,}300\,(1+i)^{n},$$

where $295,000 \{(1+i)^n - 1\} + 500\ (1.02)^{20}\ (1-v^2)\ (1+i)^n = 2,036,300\ (1+i)^n\, i$,

where $295,000\ \{(1+i)^n - 1\} + 49\ \ (1+i)^n = 71,271\ \ (1+i)^n$,

where $(1+i)^n = \dfrac{295,000}{223,778} = 1.318$, or where n is a little more than 8.

Nine years after the first harvest when the 10th harvest has just been sold, the debts should be all paid off and there should be a surplus of

$\$295,000\ s_{\overline{9|}} + \$500\ (1.02)^{20}\ s_{\overline{2|}}\ (1+i)^7 - \$2,036,300\ (1+i)^9 = \$285,361$.

And the value of the harvests to come is $\$295,000\ a_{\overline{2|}} = \$560,410$.

So that the total value when the bonds are all paid off will be \$845,771.

INTEREST AND BOND VALUES.

EXERCISES.

/ 1. What rate of interest compounded half yearly is the equivalent of 5% compounded (i) yearly (ii) quarterly?

2. Find $(1+i)^{500}$ and v^{500} at (i) 3% (ii) 5%.

3. For how long a time should \$100 be left to accumulate at 5% in order that it may amount to double the accumulated value of another \$100 deposited at the same time at 3%?

4. Find the present value of \$1000. a year for 20 years at 4% under the following conditions.

(*a*) Interest compounded (i) yearly; (ii) half yearly; (iii) quarterly.

(*b*) The \$1000 payable (i) yearly; (ii) half yearly; (iii) quarterly.

(*c*) The first payment to be made (i) now (ii) a year from now; (iii) three years hence.

5. An annuity of \$1000. a year for 25 years, first payment one year hence, is to be made payable (i) quarterly in advance. (ii) yearly in advance. (iii) half yearly, first payment 3 months hence. Find the equivalent payments in each case.

6. What payment should be made in cash by the annuitant to obtain each of the conversions of the previous question and still receive \$1000 a year?

7. A man deposited \$100 on the 1st January, 1890, and \$100 every six months thereafter, at 4% per annum compounded half-yearly, making his last deposit on the 1st January, 1910. What sum was standing to his credit immediately after his last deposit? To what will this accumulate at the same rate of interest by the 1st July, 1915?

8. A man deposited \$100 every six months at 3% compounded half yearly. Every four years he drew out the amount at his credit and invested in 5% bonds with half yearly coupons at par. The coupons of these bonds have always been deposited with the half yearly \$100. This process has been go-

ing on for 24 years and the man has just made a new purchase of bonds. How much does he now hold in bonds?

9. How would the result of the last question have been affected if the bonds had all been bought at 105?

10. Explain the following equation in words:

$\frac{1}{i} = a_{\overline{n}|} + v^n . \frac{1}{i}$, where $\frac{1}{i}$ is the value of a perpetuity of 1 per period at i per period.

11. Twenty years ago $10,000 in 4½% bonds with half yearly coupons was deposited with a trust company to accumulate as follows:—The coupons to be deposited in a savings account at 3% compounded half yearly. At the end of every fifth year the accumulated amount in the savings account to be invested in 4% bonds at par. Assuming that the Trust Company charged 5% of income, to what amount has the fund now grown?

12. Make verbal and self-explanatory statements of the following formulae.

(i) $vi=d$; (ii) $i-d=di$; (iii) $(1+i)^n=1+i\ s_{\overline{n}|}$;
(iv) $1=i\ a_{\overline{n}|}+v^n$; (v) $a^{-1}_{\overline{n}|}-s^{-1}_{\overline{n}|}=i$; (vi) $(1-v^n)\ a^{-1}_{\overline{n}|}=i$;
(vii) $\left\{(1+i)^n-1\right\} s^{-1}_{\overline{n}|}=i$.

To what do formulae (iii) to (vii) reduce when $n=1$?

13. If $x=a_{\overline{n}|}$ and $y=a_{\overline{2n}|}$ find i in terms of n. x, and y.

14. How would you express the present value of 1 due n years hence at a rate j compounded m times a year; and also the accumulated amount of 1 deposited n years ago at the same rate?

What do these expressions become when m is infinite?

15. What is the present value at 3% of an annuity to run for 25 years; the payments being $100., $103., &c., increasing 3% per annum?

16. If the payments of the annuity in the previous question were to be discounted at 5% instead of at 3%, by how much would the value differ from $\$\frac{100}{1.03}\ a_{\overline{25}|}$ at 2%?

17. Find, at 3½% interest, the value as at the 1st January, 1910, of:

(i) Twenty annual payments of \$1000 each, the last to be made on the 1st January, 1930.

(ii) Twenty annual payments of \$1000 each, the last to be made on the 1st January 1929.

(iii) A Bond for \$10,000 bearing annual coupons for interest at 4% and due on the 1st January, 1925.

18. A certain property producing a fixed income to perpetuity is left in equal shares to four hospitals, A, B, C, and D. A, B, and C are each in succession to enjoy the whole income for a time and the final reversion is to be to D. Assuming interest at 4% compounded yearly, for what length of time should A, B, and C each enjoy the income before the property goes absolutely to D ?

19. If a 30-year annuity is worth 20 years purchase, what should be paid on the same basis for a 40-year annuity?

20. Find the value of 20 future half yearly payments as follows: \$100. at the end of thirty months,
\$105. at the end of thirty-six months,
\$110. at the end of forty-two months,
and so on, increasing by \$5 each six months. Assume 5% compounded half yearly.

21. What do the following functions $(1+i)^n$, v^n, $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{-1}_{\overline{n}|}$, $s^{-1}_{\overline{n}|}$ become when

(*a*) i is zero and $n =$ (i) zero (ii) infinity,

(*b*) i is infinite and $n =$ (i) zero (ii) infinity ?

22. A man possessing a certain sum invested at rate i spends 1⅝ of his interest the first year, 3¼ times his interest the second year, 4⅞ times his interest the third year, and so on. At the end of the 16th year he has nothing left. Shew that in the 8th year he spent as much as he had left at the end of that year, and that his money was invested at 4%.

23. A man borrows \$5,000 at 5% and agrees to repay the loan by an annuity (covering principal and interest) in ten years. What annual payment will he make? How much of the fifth payment will be a return of capital? What capital will be outstanding after the fifth payment ? If the borrower then wishes to repay the balance and it is agreed to discount future pay-

ments on a 4% interest basis, what will be the redemption price?

24. An investor, calculating prices to yield him $4\frac{1}{4}$% compounded half yearly, buys $100,000 of 5% 40-year bonds with half-yearly coupons for $114,365.30. Find $a_{\overline{80|}}$ and v^{80} at $2\frac{1}{8}$%. What should that investor bid for a similar bond with yearly coupons?

25. A corporation borrows $50,000 at $4\frac{1}{2}$% and agrees to repay the loan at the end of 30 years. Assuming that it must accumulate a sinking fund at 3%, what annual charge will this loan impose upon the corporation? Had the lender been willing to accept repayment in the form of an annuity covering principal and interest at 5% what difference would it have made in the annual charge?

26. A 30-year 4% bond with half yearly coupons is bought at 95. Find the investment yield on the following suppositions:

(i) that the purchaser debits the account with interest each half year at the yield rate and credits it with the coupons as they are paid.

(ii) that the purchaser provides for the shortage in the interest each half year from another source, allowing 6% on all items so transferred.

27. A 20-year 6% bond with half yearly coupons is bought at 134. Find the investment yield on the following suppositions:

(i) that the purchaser writes his investment down each half year by the excess interest in the coupon.

(ii) that the purchaser sets up a sinking fund at 4% compounded half yearly to replace the premium at which he bought the bond.

28. Bonds to the amount of $1,000 are bought to yield $4\frac{1}{4}$% interest convertible half-yearly, reckoning from the next interest date. They bear 4% payable semi-annually, and the price paid for them is $966.50 and accrued interest of $15,00.

What are the entries required to record this purchase; also the entries when the first payment is received, assuming that the company amortizes its bonds?

29. Certain 6 per cent. bonds maturing February 1, 1934, interest payable semi-annually, contain an option giving the

right to the issuing corporation to redeem them at 110 on or after February 1, 1919. Compute the value of these bonds as at February 1, 1909, on a 5 per cent basis.

30. *A* dies, leaving an estate of \$44,000 in cash, from which a tax of 1% is to be deducted; the balance is to be invested in 5% bonds, then quoted at 132, and the income is to be divided equally among three children. What will be the annual income of each of the children on the supposition that the trustee sets up a sinking fund to replace the premium on redemption 30 years hence, and that the sinking fund will earn 4%? All interest compounded and payable half yearly.

31. Find an expression for the value of a bond due n years hence and bearing interest at the nominal rate g, payable p times a year, in order to pay the purchaser interest at the nominal rate j, convertible m times a year. What does the expression become when $m=p$?

32. A loan of \$200,000 at 5% payable half yearly is to be repaid as follows:

\$5,000 at the end of 5 years,
\$6,000 at the end of 6 years,
\$7,000 at the end of 7 years,

and so on. The issue price is 92½. What rate of interest is the borrower paying?

33. A purchases from B a piece of property worth \$10,000, agreeing to pay for it in equal instalments at the end of each year for ten years, including interest at the rate of 5% per annum. There is a tax of 1% on the property and by agreement A is to pay at the end of each year only his share of the tax, reckoning as his share 1% of the principal paid up to and including the instalment then due, but in place of a varying amount from year to year he desires to pay a level extra amount with each of the ten instalments.

Find (*a*) a general formula for the extra. (*b*) A's total annual payment, assuming 5% interest per annum throughout.

34. The value of an annuity for 30 years of which the payments are successively 30, 29, 28, etc., is 225.

Determine the interest yield.

35. (*a*) Determine an expression for the amount which should be paid a lender t years hence for an immediate advance of 1 made upon the condition that the lender is to receive interest at the rate j per annum for n years $(n>t)$, though he can make re-investments only at the rate i, $(j>i)$.

(*b*) From (*a*) obtain the present value of an annuity of 1 per annum for n years, the remunerative rate being j per annum, and the reproductive rate i per annum.

(*c*) Discuss the redemption of such an annuity after t years upon application by the borrower.

36. Investigate a convenient formula for ascertaining approximately the true rate of interest yielded by debentures terminable at the end of n years, issued at a premium and redeemable at par.

Apply the formula so obtained to determine the rate of interest yielded by a terminable 6% debenture, repayable at par at the end of 20 years, purchased at 120.

37. It is desired to raise \$100,000 by an issue of debentures, \$5,000 is to be set aside each year to pay interest and provide for the redemption of the debentures—the sum to be apportioned as follows:

(1.) Interest at 4% is to be paid at the end of each year on the debentures then outstanding.

(2.) The balance of the \$5,000 is to be invested to yield 3% to provide for triennial drawings of the debentures at a premium of 5%, the first drawing to be at the end of the third year from the date of issue.

Find the number of years necessary to pay off the loan.

38. A foreign government loan of \$1,000,000 at 5% with half yearly coupons is to be redeemed at 110 by the operation of an accumulative sinking fund in 36 years. What semi-annual sum should be set aside for the service of the loan ?

(i) When redemptions are made each half year.

(ii) When redemptions are made at intervals of 4 years from a fund which accumulates in the meantime at 4% compounded half yearly.

39. Find a ready approximation to the period in which a 5% loan with yearly coupons will be redeemed by an accumulative

sinking fund of $1\frac{1}{2}\%$ allowing for quinquennial redemptions at 110 from the sinking fund which can be invested during each quinquennium at 4% compounded half yearly.

40. A loan of $10,000 at 5% payable half yearly was made 15 years ago. For the first 5 years the borrower paid $350 each half year. For the next five years he paid $325 each half year, and for the past five years he has paid only $300 each half year. How much must he pay each half year for the next five years to extinguish the debt?

41. What payment made half yearly in advance for r years will secure a quarterly annuity of 1 per annum the first instalment of which will fall due at the commencement of the $(r+1)$th year and the last three months before the end of the $(r+n)$th year? Interest at the nominal rate i convertible quarterly throughout.

42. By paying a certain rate i per annum in quarterly instalments the effective rate becomes $(1.017)\ i$. How would you approximate to the value of i?

43. Find the value of an annuity payable annually whose several payments are 1, 2, 3, 4, etc. when the annuity is to run (i) for n years; (ii) forever.

44. If a debt bearing interest at rate i compounded yearly can be discharged, principal and interest, by n annual instalments, in how much less time would the same debt be discharged by the same annual instalments payable half yearly,

(i) When the interest on the debt remains payable yearly?

(ii) When the interest on the debt is to be payable half yearly?

45. Prove that the sum of the first x terms of the 4% values of $s_{\overline{n}|}$ is equal to $\dfrac{s_{\overline{x+1}|}-(x+1)}{.04}$

46. Shew that if a be the value of an annuity of 1 per annum payable at the end of each year, then the value of the same annuity of 1 per annum, but payable at the end of each quarter is $a\ (1+\frac{3}{8}i)$ approximately.

47. A perpetuity of $x per annum, the first payment of which is due at the end of n years, is to be purchased by an-

nual instalments, commencing at P and diminishing by $\frac{P}{n}$ each year, so that the last instalment, $\frac{P}{n}$, will be payable at the beginning of the nth year. Find the value of P.

48. An annuity of $1000 a year payable half yearly for 20 years, the first payment to be made 20 years from now, is to be purchased by ten annual instalments commencing at once and increasing by 5% per annum. Find the amount of the first instalment assuming interest at 4% compounded half yearly.

49. A man holds $10,000 in 5% municipal debentures redeemable at par in 7 years and standing in his books at $10,097. He is offered conversion into 4% inscribed stock at the rate of 111 for every 100 of his debentures. Assuming that he may count upon realising the stock at par in 7 years time, and in the meanwhile is able to invest small sums at 4%, what difference in the rate yielded by the investment would the conversion make? All rates are payable half yearly.

50. A loan of $10,000 is to be repaid in 15 years by uniform semi-annual payments which include interest at 5% for the first half of the time, and at 4½% for the second half, and also sinking fund payments which will improve at 4%. Find the semi-annual payments needed to repay this loan.

Interest Tables.

This short collection will serve the student in the solution of typical problems; but the tables given herein must not be regarded as a substitute for such tables as those of Mr. Archer or Colonel Oakes.

Table I.

Amount of 1 : *viz.*, $(1+i)^n$.

n	1%	1¼%	1½%	1¾%	2%	2¼%	n
1	1·01000	1·01250	1·01500	1·01750	1·02000	1·02250	1
2	1·02010	1·02516	1·03023	1·03531	1·04040	1·04551	2
3	1·03030	1·03797	1·04568	1·05342	1·06121	1·06903	3
4	1·04060	1·05095	1·06136	1·07186	1·08243	1·09308	4
5	1·05101	1·06408	1·07728	1·09062	1·10408	1·11768	5
6	1·06152	1·07738	1·09344	1·10970	1·12616	1·14283	6
7	1·07214	1·09085	1·10984	1·12912	1·14869	1·16854	7
8	1·08286	1·10449	1·12649	1·14888	1·17166	1·19483	8
9	1·09369	1·11829	1·14339	1·16899	1·19509	1·22171	9
10	1·10462	1·13227	1·16054	1·18944	1·21899	1·24920	10
11	1·11567	1·14642	1·17795	1·21026	1·24337	1·27731	11
12	1·12683	1·16075	1·19562	1·23144	1·26824	1·30605	12
13	1·13809	1·17526	1·21355	1·25299	1·29361	1·33544	13
14	1·14947	1·18995	1·23176	1·27492	1·31948	1·36548	14
15	1·16097	1·20483	1·25023	1·29723	1·34587	1·39621	15
16	1·17258	1·21989	1·26899	1·31993	1·37279	1·42762	16
17	1·18430	1·23514	1·28802	1·34303	1·40024	1·45974	17
18	1·19615	1·25058	1·30734	1·36653	1·42825	1·49259	18
19	1·20811	1·26621	1·32695	1·39045	1·45681	1·52617	19
20	1·22019	1·28204	1·34686	1·41478	1·48595	1·56051	20
21	1·23239	1·29806	1·36706	1·43954	1·51567	1·59562	21
22	1·24472	1·31429	1·38756	1·46473	1·54598	1·63152	22
23	1·25716	1·33072	1·40838	1·49036	1·57690	1·66823	23
24	1·26973	1·34735	1·42950	1·51644	1·60844	1·70577	24
25	1·28243	1·36419	1·45095	1·54298	1·64061	1·74415	25
26	1·29526	1·38125	1·47271	1·56998	1·67342	1·78339	26
27	1·30821	1·39851	1·49480	1·59746	1·70689	1·82352	27
28	1·32129	1·41599	1·51722	1·62541	1·74102	1·86454	28
29	1·33450	1·43369	1·53998	1·65386	1·77584	1·90650	29
30	1.34785	1·45161	1·56308	1·68280	1·81136	1·94939	30
31	1·36133	1·46976	1·58653	1·71225	1·84759	1·99325	31
32	1·37494	1·48813	1·61032	1·74221	1·88454	2·03810	32
33	1·38869	1·50673	1·63448	1·77270	1·92223	2·08396	33
34	1·40258	1·52557	1·65900	1·80372	1·96068	2·13085	34
35	1·41660	1·54464	1·68388	1·83529	1·99989	2·17879	35
36	1·43077	1·56394	1·70914	1·86741	2·03989	2·22782	36
37	1·44508	1·58349	1·73478	1·90009	2·08069	2·27794	37
38	1·45953	1·60329	1·76080	1·93334	2·12230	2·32920	38
39	1·47412	1·62333	1·78721	1·96717	2·16474	2·38160	39
40	1·48886	1·64362	1·81402	2·00160	2·20804	2·43519	40
41	1·50375	1·66416	1·84123	2·03663	2·25220	2·48998	41
42	1·51879	1·68497	1·86885	2·07227	2·29724	2·54601	42
43	1·53398	1·70603	1·89688	2·10853	2·34319	2·60329	43
44	1.54932	1·72735	1·92533	2·14543	2·39005	2·66186	44
45	1·56481	1·74895	1·95421	2·18298	2·43785	2·72176	45
46	1·58046	1·77081	1·98353	2·22118	2·48661	2·78300	46
47	1·59626	1·79294	2·01328	2·26005	2·53634	2·84561	47
48	1·61223	1·81535	2·04348	2·29960	2·58707	2·90964	48
49	1·62835	1·83805	2·07413	2·33984	2·63881	2·97511	49
50	1·64463	1·86102	2·10524	2·38079	2·69159	3·04205	50

TABLE I. (*continued*).

Amount of 1 : *viz.*, $(1+i)^n$.

n	2½%	3%	3½%	4%	4½%	5%	n
1	1·02500	1·03000	1·03500	1·04000	1·04500	1·05000	1
2	1·05063	1·06090	1·07123	1·08160	1·09203	1·10250	2
3	1.07689	1·09273	1·10872	1·12486	1·14117	1·15763	3
4	1·10381	1·12551	1·14752	1·16986	1·19252	1·21551	4
5	1·13141	1·15927	1·18769	1·21665	1·24618	1·27628	5
6	1·15969	1·19405	1·22926	1·26532	1·30226	1·34010	6
7	1·18869	1·22987	1·27228	1·31593	1·36086	1·40710	7
8	1·21840	1·26677	1·31681	1·36857	1·42210	1·47746	8
9	1·24886	1·30477	1·36290	1·42331	1·48610	1·55133	9
10	1·28008	1·34392	1·41060	1·48024	1·55297	1·62889	10
11	1·31209	1·38423	1·45997	1·53945	1·62285	1·71034	11
12	1·34489	1·42576	1·51107	1·60103	1·69588	1·79586	12
13	1·37851	1·46853	1·56396	1·66507	1·77220	1·88565	13
14	1·41297	1·51259	1·61869	1·73168	1·85194	1·97993	14
15	1·44830	1·55797	1·67535	1·80094	1·93528	2·07893	15
16	1·48451	1·60471	1·73399	1·87298	2·02237	2·18287	16
17	1·52162	1·65285	1·79468	1·94790	2·11338	2·29202	17
18	1·55966	1·70243	1·85749	2·02582	2·20848	2·40662	18
19	1·59865	1·75351	1·92250	2·10685	2·30786	2·52695	19
20	1·63862	1·80611	1·98979	2·19112	2·41171	2·65330	20
21	1·67958	1·86029	2·05943	2·27877	2·52024	2·78596	21
22	1·72157	1·91610	2·13151	2·36992	2·63365	2·92526	22
23	1·76461	1·97359	2·20611	2·46472	2·75217	3·07152	23
24	1·80873	2·03279	2·28333	2·56330	2·87601	3·22510	24
25	1·85394	2·09378	2·36324	2·66584	3·00543	3·38635	25
26	1·90029	2·15659	2·44596	2·77247	3·14068	3·55567	26
27	1·94780	2·22129	2·53157	2·88337	3·28201	3·73346	27
28	1·99650	2.28793	2·62017	2·99870	3·42970	3·92013	28
29	2·04641	2·35657	2·71188	3·11865	3·58404	4·11614	29
30	2·09757	2·42726	2·80679	3·24340	3·74532	4·32194	30
31	2·15001	2·50008	2·90503	3·37313	3·91386	4·53804	31
32	2·20376	2·57508	3·00671	3·50806	4·08998	4·76494	32
33	2·25885	2·65234	3·11194	3·64838	4·27403	5·00319	33
34	2·31532	2·73191	3·22086	3·79432	4·46636	5·25335	34
35	2·37321	2·81386	3·33359	3·94609	4·66735	5·51602	35
36	2·43254	2·89828	3·45027	4·10393	4·87738	5·79182	36
37	2·49335	2·98523	3·57103	4·26809	5·09686	6·08141	37
38	2·55568	3·07478	3·69601	4·43881	5·32622	6·38548	38
39	2·61957	3·16703	3·82537	4·61637	5·56590	6·70475	39
40	2·68506	3·26204	3·95926	4·80102	5·81636	7·03999	40
41	2·75219	3·35990	4·09783	4·99306	6·07810	7·39199	41
42	2·82100	3·46070	4·24126	5·19278	6·35162	7·76159	42
43	2·89152	3·56452	4·38970	5·40050	6·63744	8·14967	43
44	2·96381	3·67145	4·54334	5·61652	6·93612	8·55715	44
45	3·03790	3·78160	4·70236	5·84118	7·24825	8·98501	45
46	3·11385	3·89504	4·86694	6·07482	7·57442	9·43426	46
47	3·19170	4·01190	5·03728	6·31782	7·91527	9·90597	47
48	3·27149	4·13225	5·21359	6·57053	8·27146	10·40127	48
49	3·35328	4·25622	5·39606	6·83335	8·64367	10·92133	49
50	3·43711	4·38391	5·58493	7·10668	9·03264	11·46740	50

Table II.

Present Value of 1 : *viz.*, v^n.

n	1%	1¼%	1½%	1¾%	2%	2¼%	n
1	·99010	·98765	·98522	·98280	·98039	·97800	1
2	·98030	·97546	·97066	·96590	·96117	·95647	2
3	·97059	·96342	·95632	·94929	·94232	·93543	3
4	·96098	·95152	·94218	·93296	·92385	·91484	4
5	·95147	·93978	·92826	·91691	·90573	·89471	5
6	·94205	·92817	·91454	·90114	·88797	·87502	6
7	·93272	·91672	·90103	·88564	·87056	·85577	7
8	·92348	·90540	·88771	·87041	·85349	·83694	8
9	·91434	·89422	·87459	·85544	·83676	·81852	9
10	·90529	·88318	·86167	·84073	·82035	·80051	10
11	·89632	·87228	·84893	·82627	·80426	·78289	11
12	·88745	·86151	·83639	·81206	·78849	·76567	12
13	·87866	·85087	·82403	·79809	·77303	·74882	13
14	·86996	·84037	·81185	·78436	·75788	·73234	14
15	·86135	·82999	·79985	·77087	·74301	·71623	15
16	·85282	·81975	·78803	·75762	·72845	·70047	16
17	·84438	·80963	·77639	·74459	·71416	·68505	17
18	·83602	·79963	·76491	·73178	·70016	.66998	18
19	·82774	·78976	·75361	·71919	·68643	·65523	19
20	·81954	·78001	·74247	·70682	·67297	·64082	20
21	·81143	·77038	·73150	·69467	·65978	·62672	21
22	·80340	·76087	·72069	·68272	·64684	·61292	22
23	·79544	·75147	·71004	·67098	·63416	·59944	23
24	·78757	·74220	·69954	·65944	·62172	·58625	24
25	·77977	·73303	·68921	·64810	·60953	·57335	25
26	·77205	·72398	·67902	·63695	·59758	·56073	26
27	·76440	·71505	·66899	·62599	58586	·54839	27
28	·75684	·70622	·65910	·61523	·57437	·53632	28
29	·74934	·69750	·64936	·60465	·56311	·52452	29
30	·74192	·68889	·63976	·59425	·55207	·51298	30
31	·73458	·68038	·63031	·58403	·54125	·50169	31
32	·72730	·67198	·62099	·57398	·53063	·49065	32
33	·72010	·66369	·61182	·56411	·52023	·47986	33
34	·71297	·65549	·60277	·55441	·51003	·46930	34
35	·70591	·64740	·59387	·54487	·50003	·45897	35
36	·69892	·63941	·58509	·53550	·49022	·44887	36
37	·69200	·63152	·57644	·52629	·48061	·43899	37
38	·68515	·62372	·56792	·51724	·47119	·42933	38
39	·67837	·61602	·55953	·50834	·46195	·41989	39
40	·67165	·60841	·55126	·49960	·45289	·41065	40
41	·66500	·60090	·54312	·49101	·44401	·40161	41
42	·65842	·59348	·53509	·48256	·43530	·39277	42
43	·65190	·58616	·52718	·47426	·42677	·38413	43
44	·64545	·57892	·51939	·46611	·41840	·37568	44
45	·63905	·57177	·51171	·45809	41020	·36741	45
46	·63273	·56471	·50415	·45021	·40215	·35932	46
47	·62646	·55774	·49670	·44247	·39427	·35142	47
48	·62026	·55086	·48936	·43486	·38654	·34369	48
49	·61412	·54406	·48213	·42738	·37896	·33612	49
50	·60804	·53734	·47500	·42003	·37153	·32873	50

Table II. (*continued*).

Present Value of 1 : *viz.*, v^n.

n	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	n
1	·97561	·97087	·96618	·96154	·95694	·95238	1
2	·95181	·94260	·93351	·92456	·91573	·90703	2
3	·92860	·91514	·90194	·88900	·87630	·86384	3
4	·90595	·88849	·87144	·85480	·83856	·82270	4
5	·88385	·86261	·84197	·82193	·80245	·78353	5
6	·86230	·83748	·81350	·79031	·76790	·74622	6
7	·84127	·81309	·78599	·75992	·73483	·71068	7
8	·82075	·78941	·75941	·73069	·70319	·67684	8
9	·80073	·76642	·73373	·70259	·67290	·64461	9
10	·78120	·74409	·70892	·67556	·64393	·61391	10
11	·76214	·72242	·68495	·64958	·61620	·58468	11
12	·74356	·70138	·66178	·62460	·58966	·55684	12
13	·72542	·68095	·63940	·60057	·56427	·53032	13
14	·70773	·66112	·61778	·57748	·53997	·50507	14
15	·69047	·64186	·59689	·55526	·51672	·48102	15
16	·67362	·62317	·57671	·53391	·49447	·45811	16
17	·65720	·60502	·55720	·51337	·47318	·43630	17
18	·64117	·58739	·53836	·49363	·45280	·41552	18
19	·62553	·57029	·52016	·47464	·43330	·39573	19
20	·61027	·55368	·50257	·45639	·41464	·37689	20
21	·59539	·53755	·48557	·43883	·39679	·35894	21
22	·58086	·52189	·46915	·42196	·37970	·34185	22
23	·56670	·50669	·45329	·40573	·36335	·32557	23
24	·55288	·49193	·43796	·39012	·34770	·31007	24
25	·53939	·47761	·42315	·37512	·33273	·29530	25
26	·52623	·46369	·40884	·36069	·31840	·28124	26
27	·51340	·45019	·39501	·34682	·30469	·26785	27
28	·50088	·43708	·38165	·33348	·29157	·25509	28
29	·48866	·42435	·36875	·32065	·27902	·24295	29
30	·47674	·41199	·35628	·30832	·26700	·23138	30
31	·46511	·39999	·34423	·29646	·25550	·22036	31
32	·45377	·38834	·33259	·28506	·24450	·20987	32
33	·44270	·37703	·32134	·27409	·23397	·19987	33
34	·43191	·36604	·31048	·26355	·22390	·19035	34
35	·42137	·35538	·29998	·25342	·21425	·18129	35
36	·41109	·34503	·28983	·24367	·20503	·17266	36
37	·40107	·33498	·28003	·23430	·19620	·16444	37
38	·39128	·32523	·27056	·22529	·18775	·15661	38
39	·38174	·31575	·26141	·21662	·17967	·14915	39
40	·37243	·30656	·25257	·20829	·17193	·14205	40
41	·36335	·29763	·24403	·20028	·16453	·13528	41
42	·35448	·28896	·23578	·19257	·15744	·12884	42
43	·34584	·28054	·22781	·18517	·15066	·12270	43
44	·33740	·27237	·22010	·17805	·14417	·11686	44
45	·32917	·26444	·21266	·17120	·13796	·11130	45
46	·32115	·25674	·20547	·16461	·13202	·10600	46
47	·31331	·24926	·19852	·15828	·12634	·10095	47
48	·30567	·24200	·19181	·15219	·12090	·09614	48
49	·29822	·23495	·18532	·14634	·11569	·09156	49
50	·29094	·22811	·17905	·14071	·11071	·08720	50

Table III.

Amount of 1 *per Period: viz.*, $s_{\overline{n}|}$.

n	1%	1¼%	1½%	1¾%	2%	2¼%	*n*
1	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	1
2	2·01000	2.01250	2·01500	2·01750	2·02000	2·02250	2
3	3·03010	3·03766	3·04523	3·05281	3·06040	3·06801	3
4	4·06040	4·07563	4·09090	4·10623	4·12161	4·13704	4
5	5·10101	5·12657	5·15227	5·17809	5·20404	5·23012	5
6	6·15202	6·19065	6·22955	6·26871	6·30812	6·34780	6
7	7·21354	7·26804	7·32299	7·37841	7·43428	7·49062	7
8	8·28567	8·35889	8·43284	8·50753	8·58297	8·65916	8
9	9·36853	9·46337	9·55933	9·65641	9·75463	9·85399	9
10	10·46221	10·58167	10·70272	10·82540	10·94972	11·07571	10
11	11·56683	11·71394	11·86326	12·01484	12·16872	12·32491	11
12	12·68250	12·86036	13·04121	13·22510	13·41209	13·60222	12
13	13·80933	14·02112	14·23683	14·45654	14·68033	14·90827	13
14	14·94742	15·19638	15·45038	15·70953	15·97394	16·24371	14
15	16·09690	16·38633	16·68214	16·98445	17·29342	17·60919	15
16	17·25786	17·59116	17·93237	18·28168	18·63929	19·00540	16
17	18·43044	18·81105	19·20136	19·60161	20·01207	20·43302	17
18	19·61475	20·04619	20.48938	20·94463	21·41231	21·89276	18
19	20·81089	21·29677	21·79672	22·31117	22·84056	23·38535	19
20	22·01900	22·56298	23·12367	23·70161	24·29737	24·91152	20
21	23·23919	23·84502	24·47052	25·11639	25·78332	26.47203	21
22	24·47159	25·14308	25·83758	26·55593	27·29898	28·06765	22
23	25·71630	26·45737	27·22514	28·02065	28·84496	29·69917	23
24	26·97346	27·78808	28·63352	29·51102	30·42186	31·36740	24
25	28·24320	29·13544	30·06302	31·02746	32·03030	33·07317	25
26	29·52563	30·49963	31·51397	32·57044	33·67091	34·81732	26
27	30·82089	31·88087	32·98668	34·14042	35·34432	36·60071	27
28	32·12910	33·27938	34·48148	35·73788	37·05121	38·42422	28
29	33·45039	34·69538	35·99870	37·36329	38·79223	40·28877	29
30	34·78489	36·12907	37·53868	39·01715	40·56808	42·19526	30
31	36·13274	37·58068	39·10176	40·69995	42·37944	44·14466	31
32	37·49407	39·05044	40·68829	42·41220	44·22703	46·13791	32
33	38·86901	40·53857	42·29861	44·15441	46·11157	48·17602	33
34	40·25770	42·04530	43·93309	45·92712	48·03380	50·25998	34
35	41·66028	43·57087	45·59209	47·73084	49·99448	52·39083	35
36	43·07688	45·11551	47·27597	49·56613	51·99437	54·56962	36
37	44·50765	46·67945	48·98511	51·43354	54·03425	56·79744	37
38	45·95272	48·25294	50·71989	53·33362	56·11494	59·07538	38
39	47·41225	49·86623	52·48068	55·26696	58·23724	61·40457	39
40	48·88637	51·48956	54·26789	57·23413	60·40198	63·78618	40
41	50·37524	53·13318	56·08191	59·23573	62·61002	66·22137	41
42	51·87899	54·79734	57·92314	61·27236	64·86222	68·71135	42
43	53·39778	56·48231	59·79199	63·34462	67·15947	71·25735	43
44	54·93176	58·18834	61·68887	65·45315	69·50266	73·86064	44
45	56·48107	59·91569	63·61420	67·59858	71·89271	76·52251	45
46	58·04588	61·66464	65·56841	69·78156	74·33056	79·24426	46
47	59·62634	63·43545	67·55194	72·00274	76·81718	82·02726	47
48	61·22261	65·22839	69·56522	74·26278	79·35352	84·87287	48
49	62·83483	67·04374	71·60870	76·56238	81·94059	87·78251	49
50	64·46318	68·88179	73·68283	78·90222	84·57940	90·75762	50

Table III. (*continued*).

Amount of 1 *per Period: viz.*, $s_{\overline{n}|}$.

n	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	n
1	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1
2	2·0250	2·0300	2·0350	2·0400	2·0450	2·0500	2
3	3·0756	3·0909	3·1062	3·1216	3·1370	3·1525	3
4	4·1525	4·1836	4·2149	4·2465	4·2782	4·3101	4
5	5·2563	5·3091	5·3625	5·4163	5·4707	5·5256	5
6	6·3877	6·4684	6·5502	6·6330	6·7169	6·8019	6
7	7·5474	7·6625	7·7794	7·8983	8·0192	8·1420	7
8	8·7361	8·8923	9·0517	9·2142	9·3800	9·5491	8
9	9·9545	10·1591	10·3685	10·5828	10·8021	11·0266	9
10	11·2034	11·4639	11·7314	12·0061	12·2882	12·5779	10
11	12·4835	12·8078	13·1420	13·4864	13·8412	14·2068	11
12	13·7956	14·1920	14·6020	15·0258	15·4640	15·9171	12
13	15·1404	15·6178	16·1130	16·6268	17·1599	17·7130	13
14	16·5190	17·0863	17·6770	18·2919	18·9321	19·5986	14
15	17·9319	18·5989	19·2957	20·0236	20·7841	21·5786	15
16	19·3802	20·1569	20·9710	21·8245	22·7193	23·6575	16
17	20·8647	21·7616	22·7050	23·6975	24·7417	25·8404	17
18	22·3863	23·4144	24·4997	25·6454	26·8551	28·1324	18
19	23·9460	25·1169	26·3572	27·6712	29·0636	30·5390	19
20	25·5447	26·8704	28·2797	29·7781	31·3714	33·0660	20
21	27·1833	28·6765	30·2695	31·9692	33·7831	35·7193	21
22	28·8629	30·5368	32·3289	34·2480	36·3034	38·5052	22
23	30·5844	32·4529	34·4604	36·6179	38·9370	41·4305	23
24	32·3490	34·4265	36·6665	39·0826	41·6892	44·5020	24
25	34·1578	36·4593	38·9499	41·6459	44·5652	47·7271	25
26	36·0117	38·5530	41·3131	44·3117	47·5706	51·1135	26
27	37·9120	40·7096	43·7591	47·0842	50·7113	54·6691	27
28	39·8598	42·9309	46·2906	49·9676	53·9933	58·4026	28
29	41·8563	45·2189	48·9108	52·9663	57·4230	62·3227	29
30	43·9027	47·5754	51·6227	56·0849	61·0071	66·4388	30
31	46·0003	50·0027	54·4295	59·3283	64·7524	70·7608	31
32	48·1503	52·5028	57·3345	62·7015	68·6662	75·2988	32
33	50·3540	55·0778	60·3412	66·2095	72·7562	80·0638	33
34	52·6129	57·7302	63·4532	69·8579	77·0303	85·0670	34
35	54·9282	60·4621	66·6740	73·6522	81·4966	90·3203	35
36	57·3014	63·2759	70·0076	77·5983	86·1640	95·8363	36
37	59·7339	66·1742	73·4579	81·7022	91·0413	101·6281	37
38	62·2273	69·1594	77·0289	85·9703	96·1382	107·7095	38
39	64·7830	72·2342	80·7249	90·4091	101·4644	114·0950	39
40	67·4026	75·4013	84·5503	95·0255	107·0303	120·7998	40
41	70·0876	78·6633	88·5095	99·8265	112·8467	127·8398	41
42	72·8398	82·0232	92·6074	104·8196	118·9248	135·2318	42
43	75·6608	85·4839	96·8486	110·0124	125·2764	142·9933	43
44	78·5523	89·0484	101·2383	115·4129	131·9138	151·1430	44
45	81·5161	92·7199	105·7817	121·0294	138·8500	159·7002	45
46	84·5540	96·5015	110·4840	126·8706	146·0982	168·6852	46
47	87·6679	100·3965	115·3510	132·9454	153·6726	178·1194	47
48	90·8596	104·4084	120·3883	139·2632	161·5879	188·0254	48
49	94·1311	108·5406	125·6018	145·8337	169·8594	198·4267	49
50	97·4843	112·7969	130·9979	152·6671	178·5030	209·3480	50

Table IV.

Present Value of 1 *per Period: viz.,* $a_{\overline{n}|}$.

n	1%	1¼%	1½%	1¾%	2%	2¼%	*n*
1	0·99010	0·98765	0·98522	0·98280	0·98039	0·97800	1
2	1·97040	1·96312	1·95588	1·94870	1·94156	1·93447	2
3	2·94099	2·92653	2·91220	2·89798	2·88388	2·86990	3
4	3·90197	3·87806	3·85438	3·83094	3·80773	3·78474	4
5	4·85343	4·81783	4·78265	4·74786	4·71346	4·67945	5
6	5·79548	5·74601	5·69719	5·64900	5·60143	5·55448	6
7	6·72819	6·66273	6·59821	6·53464	6·47199	6·41025	7
8	7·65168	7·56812	7·48593	7·40505	7·32548	7·24718	8
9	8·56602	8·46234	8·36052	8·26049	8·16224	8·06571	9
10	9·47130	9·34553	9·22219	9·10122	8·98258	8·86622	10
11	10·36763	10·21780	10·07112	9·92749	9·78685	9·64911	11
12	11·25508	11·07931	10·90751	10·73955	10·57534	10·41478	12
13	12·13374	11·93018	11·73153	11·53764	11·34837	11·16360	13
14	13·00370	12·77055	12·54338	12·32201	12·10625	11·89594	14
15	13·86505	13·60055	13·34323	13·09288	12·84926	12·61217	15
16	14·71787	14·42029	14·13126	13·85050	13·57771	13·31263	16
17	15·56225	15·22992	14·90765	14·59508	14·29187	13·99768	17
18	16·39827	16·02955	15·67256	15·32686	14·99203	14·66766	18
19	17·22601	16·81931	16·42617	16·04606	15·67846	15·32290	19
20	18·04555	17·59932	17·16864	16·75288	16·35143	15·96371	20
21	18·85698	18·36969	17·90014	17·44755	17·01121	16·59043	21
22	19·66038	19·13056	18·62083	18·13027	17·65805	17·20335	22
23	20·45582	19·88204	19·33086	18·80125	18·29220	17·80279	23
24	21·24339	20·62423	20·03041	19·46069	18·91393	18·38904	24
25	22·02316	21·35727	20·71961	20·10878	19·52346	18·96238	25
26	22·79520	22·08125	21·39863	20·74573	20·12104	19·52311	26
27	23·55961	22·79630	22·06762	21·37173	20·70690	20·07150	27
28	24·31644	23·50252	22·72672	21·98695	21·28127	20·60783	28
29	25·06579	24·20002	23·37608	22·59160	21·84438	21·13235	29
30	25·80771	24·88891	24·01584	23·18585	22·39646	21·64533	30
31	26·54229	25·56929	24·64615	23·76988	22·93770	22·14702	31
32	27·26959	26·24127	25·26714	24·34386	23·46833	22·63767	32
33	27·98969	26·90496	25·87896	24·90797	23·98856	23·11753	33
34	28·70267	27·56046	26·48173	25·46238	24·49859	23·58683	34
35	29·40858	28·20786	27·07560	26·00725	24·99862	24·04580	35
36	30·10751	28·84727	27·66068	26·54275	25·48884	24·49467	36
37	30·79951	29·47878	28·23713	27·06904	25·96945	24·93366	37
38	31·48466	30·10250	28·80505	27·58628	26·44064	25·36299	38
39	32·16303	30·71852	29·36458	28·09463	26·90259	25·78288	39
40	32·83469	31·32693	29·91585	28·59423	27·35548	26·19352	40
41	33·49969	31·92784	30·45896	29·08524	27·79949	26·59513	41
42	34·15811	32·52132	30·99405	29·56780	28·23479	26·98790	42
43	34·81001	33·10748	31·52123	30·04207	28·66156	27·37203	43
44	35·45545	33·68640	32·04062	30·50817	29·07996	27·74771	44
45	36·09451	34·25817	32·55234	30·96626	29·49016	28·11512	45
46	36·72724	34·82288	33·05649	31·41647	29·89231	28·47444	46
47	37·35370	35·38062	33·55319	31·85894	30·28658	28·82586	47
48	37·97396	35·93148	34·04255	32·29380	30·67312	29·16955	48
49	38·58808	36·47554	34·52468	32·72118	31·05208	29·50567	49
50	39·19612	37·01288	34·99969	33·14121	31·42361	29·83440	50

Table IV. (*continued*).

Present Value of 1 *per Period: viz.,* $a_{\overline{n}|}$.

n	2½%	3%	3½%	4%	4½%	5%	n
1	0·9756	0·9709	0·9662	0·9615	0·9569	0·9524	1
2	1·9274	1·9135	1·8997	1·8861	1·8727	1·8594	2
3	2·8560	2·8286	2·8016	2·7751	2·7490	2·7232	3
4	3·7620	3·7171	3·6731	3·6299	3·5875	3·5460	4
5	4·6458	4·5797	4·5151	4·4518	4·3900	4·3295	5
6	5·5081	5·4172	5·3286	5·2421	5·1579	5·0757	6
7	6·3494	6·2303	6·1145	6·0021	5·8927	5·7864	7
8	7·1701	7·0197	6·8740	6·7327	6·5959	6·4632	8
9	7·9709	7·7861	7·6077	7·4353	7·2688	7·1078	9
10	8·7521	8·5302	8·3166	8·1109	7·9127	7·7217	10
11	9·5142	9·2526	9·0016	8·7605	8·5289	8·3064	11
12	10·2578	9·9540	9·6633	9·3851	9·1186	8·8633	12
13	10·9832	10·6350	10·3027	9·9856	9·6829	9·3936	13
14	11·6909	11·2961	10·9205	10·5631	10·2228	9·8986	14
15	12·3814	11·9379	11·5174	11·1184	10·7395	10·3797	15
16	13·0550	12·5611	12·0941	11·6523	11·2340	10·8378	16
17	13·7122	13·1661	12·6513	12·1657	11·7072	11·2741	17
18	14·3534	13·7535	13·1897	12·6593	12·1600	11·6896	18
19	14·9789	14·3238	13·7098	13·1339	12·5933	12·0853	19
20	15·5892	14·8775	14·2124	13·5903	13·0079	12·4622	20
21	16·1845	15·4150	14·6980	14·0292	13·4047	12·8212	21
22	16·7654	15·9369	15·1671	14·4511	13·7844	13·1630	22
23	17·3321	16·4436	15·6204	14·8568	14·1478	13·4886	23
24	17·8850	16·9355	16·0584	15·2470	14·4955	13·7986	24
25	18·4244	17·4131	16·4815	15·6221	14·8282	14·0939	25
26	18·9506	17·8768	16·8904	15·9828	15·1466	14·3752	26
27	19·4640	18·3270	17·2854	16·3296	15·4513	14·6430	27
28	19·9649	18·7641	17·6670	16·6631	15·7429	14·8981	28
29	20·4535	19·1885	18·0358	16·9837	16·0219	15·1411	29
30	20·9303	19·6004	18·3920	17·2920	16·2889	15·3725	30
31	21·3954	20·0004	18·7363	17·5885	16·5444	15·5928	31
32	21·8492	20·3888	19·0689	17·8736	16·7889	15·8027	32
33	22·2919	20·7658	19·3902	18·1476	17·0229	16·0025	33
34	22·7238	21·1318	19·7007	18·4112	17·2468	16·1929	34
35	23·1452	21·4872	20·0007	18·6646	17·4610	16·3742	35
36	23·5563	21·8323	20·2905	18·9083	17·6660	16·5469	36
37	23·9573	22·1672	20·5705	19·1426	17·8622	16·7113	37
38	24·3486	22·4925	20·8411	19·3679	18·0500	16·8679	38
39	24·7303	22·8082	21·1025	19·5845	18·2297	17·0170	39
40	25·1028	23·1148	21·3551	19·7928	18·4016	17·1591	40
41	25·4661	23·4124	21·5991	19·9931	18·5661	17·2944	41
42	25·8206	23·7014	21·8349	20·1856	18·7235	17·4232	42
43	26·1664	23·9819	22·0627	20·3708	18·8742	17·5459	43
44	26·5038	24·2543	22·2828	20·5488	19·0184	17·6628	44
45	26·8330	24·5187	22·4955	20·7200	19·1563	17·7741	45
46	27·1542	24·7754	22·7009	20·8847	19·2884	17·8801	46
47	27·4675	25·0247	22·8994	21·0429	19·4147	17·9810	47
48	27·7732	25·2667	23·0912	21·1951	19·5356	18·0772	48
49	28·0714	25·5017	23·2766	21·3415	19·6513	18·1687	49
50	28·3623	25·7298	23·4556	21·4822	19·7620	18·2559	50

Table V.

Periodical Payment that 1 *will purchase: viz.*, $a_{\overline{n}|}^{-1}$.

n	1%	1¼%	1½%	1¾%	2%	2¼%	n
1	1·010000	1·012500	1·015000	1·017500	1·020000	1·022500	1
2	0·507512	0·509394	0·511278	0·513163	0·515050	0·516938	2
3	·340022	·341701	·343383	·345067	·346755	·348445	3
4	·256281	·257861	·259445	·261032	·262624	·264219	4
5	·206040	·207562	·209089	·210621	·212158	·213700	5
6	·172548	·174034	·175525	·177023	·178526	·180035	6
7	·148628	·150089	·151556	·153031	·154512	·156000	7
8	·130690	·132133	·133584	·135043	·136510	·137985	8
9	·116740	·118171	·119610	·121058	·122515	·123982	9
10	·105582	·107003	·108434	·109875	·111327	·112788	10
11	·096454	·097868	·099294	·100730	·102178	·103636	11
12	·088849	·090258	·091680	·093114	·094560	·096017	12
13	·082415	·083821	·085240	·086673	·088118	·089577	13
14	·076901	·078305	·079723	·081156	·082602	·084062	14
15	·072124	·073526	·074944	·076377	·077825	·079289	15
16	·067945	·069347	·070765	·072200	·073650	·075117	16
17	·064258	·065660	·067080	·068516	·069970	·071440	17
18	·060982	·062385	·063806	·065245	·066702	·068177	18
19	·058052	·059455	·060878	·062321	·063782	·065262	19
20	·055415	·056820	·058246	·059691	·061157	·062642	20
21	·053031	·054437	·055866	·057315	·058785	·060276	21
22	·050864	·052272	·053703	·055156	·056631	·058128	22
23	·048886	·050297	·051731	·053188	·054668	·056171	23
24	·047073	·048487	·049924	·051386	·052871	·054380	24
25	·045407	·046822	·048263	·049730	·051220	·052736	25
26	·043869	·045287	·046732	·048203	·049699	·051221	26
27	·042446	·043867	·045315	·046791	·048293	·049822	27
28	·041124	·042549	·044001	·045482	·046990	·048525	28
29	·039895	·041322	·042779	·044264	·045778	·047321	29
30	·038748	·040179	·041639	·043130	·044650	·046199	30
31	·037676	·039109	·040574	·042070	·043596	·045153	31
32	·036671	·038108	·039577	·041078	·042611	·044174	32
33	·035727	·037168	·038641	·040148	·041687	·043257	33
34	·034840	·036284	·037762	·039274	·040819	·042397	34
35	·034004	·035451	·036934	·038451	·040002	·041587	35
36	·033214	·034665	·036152	·037675	·039233	·040825	36
37	·032468	·033923	·035414	·036943	·038507	·040106	37
38	·031762	·033220	·034716	·036250	·037821	·039428	38
39	·031092	·032554	·034055	·035594	·037171	·038785	39
40	·030456	·031921	·033427	·034972	·036556	·038177	40
41	·029851	·031321	·032831	·034382	·035972	·037601	41
42	·029276	·030749	·032264	·033821	·035417	·037054	42
43	·028727	·030205	·031725	·033287	·034890	·036534	43
44	·028204	·029686	·031210	·032778	·034388	·036039	44
45	·027705	·029190	·030720	·032293	·033910	·035568	45
46	·027228	·028717	·030251	·031830	·033453	·035119	46
47	·026771	·028264	·029803	·031388	·033018	·034691	47
48	·026334	·027831	·029375	·030966	·032602	·034282	48
49	·025915	·027416	·028965	·030561	·032204	·033892	49
50	·025513	·027018	·028572	·030174	·031823	·033518	50

TABLE V. (*continued*).

Periodical Payment that 1 *will purchase: viz.*, $a_{\overline{n}|}^{-1}$.

n	$2\frac{1}{2}\%$	3%	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	n
1	1·025000	1·030000	1·035000	1·040000	1·045000	1·050000	1
2	0·518827	0·522611	0·526400	0·530196	0·533998	0·537805	2
3	·350137	·353530	·356934	·360349	·363773	·367209	3
4	·265818	·269027	·272251	·275490	·278744	·282012	4
5	·215247	·218355	·221481	·224627	·227792	·230975	5
6	·181550	·184598	·187668	·190762	·193878	·197017	6
7	·157495	·160506	·163544	·166610	·169701	·172820	7
8	·139467	·142456	·145477	·148528	·151610	·154722	8
9	·125457	·128434	·131446	·134493	·137574	·140690	9
10	·114259	·117231	·120241	·123291	·126379	·129505	10
11	·105106	·108077	·111092	·114149	·117248	·120389	11
12	·097487	·100462	·103484	·106552	·109666	·112825	12
13	·091048	·094030	·097062	·100144	·103275	·106456	13
14	·085537	·088526	·091571	·094669	·097820	·101024	14
15	·080766	·083767	·086825	·089941	·093114	·096342	15
16	·076599	·079611	·082685	·085820	·089015	·092270	16
17	·072928	·075953	·079043	·082199	·085418	·088699	17
18	·069670	·072709	·075817	·078993	·082237	·085546	18
19	·066761	·069814	·072940	·076139	·079407	·082745	19
20	·064147	·067216	·070361	·073582	·076876	·080243	20
21	·061787	·064872	·068037	·071280	·074601	·077996	21
22	·059647	·062747	·065932	·069199	·072546	·075971	22
23	·057696	·060814	·064019	·067309	·070682	·074137	23
24	·055913	·059047	·062273	·065587	·068987	·072471	24
25	·054276	·057428	·060674	·064012	·067439	·070952	25
26	·052769	·055938	·059205	·062567	·066021	·069564	26
27	·051377	·054564	·057852	·061239	·064719	·068292	27
28	·050088	·053293	·056603	·060013	·063521	·067123	28
29	·048891	·052115	·055445	·058880	·062415	·066046	29
30	·047778	·051019	·054371	·057830	·061392	·065051	30
31	·046739	·049999	·053372	·056855	·060443	·064132	31
32	·045768	·049047	·052442	·055949	·059563	·063280	32
33	·044859	·048156	·051572	·055104	·058745	·062490	33
34	·044007	·047322	·050760	·054315	·057982	·061755	34
35	·043206	·046539	·049998	·053577	·057270	·061072	35
36	·042452	·045804	·049284	·052887	·056606	·060434	36
37	·041741	·045112	·048613	·052240	·055984	·059840	37
38	·041070	·044459	·047982	·051632	·055402	·059284	38
39	·040436	·043844	·047388	·051061	·054856	·058765	39
40	·039836	·043262	·046827	·050523	·054343	·058278	40
41	·039268	·042712	·046298	·050017	·053862	·057822	41
42	·038729	·042192	·045798	·049540	·053409	·057395	42
43	·038217	·041698	·045325	·049090	·052982	·056993	43
44	·037730	·041230	·044878	·048665	·052581	·056616	44
45	·037268	·040785	·044453	·048262	·052202	·056262	45
46	·036827	·040363	·044051	·047882	·051845	·055928	46
47	·036407	·039961	·043669	·047522	·051507	·055614	47
48	·036006	·039578	·043306	·047181	·051189	·055318	48
49	·035623	·039213	·042962	·046857	·050887	·055040	49
50	·035258	·038865	·042634	·046550	·050602	·054777	50